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Engineering Mechanics I

Phongsaen PITAKWATCHARA
Phongsaen.P@chula.ac.th

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Preface

This document is a supplement to the powerpoint presentation of **"2103-213 Engineering Mechanics I"**. It contains the explanation of the materials covered in the class. In addition, selected problems with solutions have been included to illustrate the application of the related subjects. Though self-contained, it does not intend to be the substitute of textbooks such as [2], [1], [4], [3], [5], and [6]; upon them the materials and problems are based. Instead, it is supposed to assist the students in taking notes and reviewing the subject.

Although I have worked my best to prepare and revise the document, some errors might have been uncaught and some explanation may be unclear. Therefore, I will be very grateful for any notice or comment which will help improving the materials. Lastly, I hope the students and readers will find it useful for their studies and careers.

Chulalongkorn University P P T
April 2011

Chapter 1

Introduction to Statics

1.1 Basic Concepts

In this section, we roughly explain the basic technical terms that will be used throughout the course.

- **Space** is the region occupied by the bodies. We set up an *coordinate system* to specify where the object is by the *position* and its posture by the *orientation*.
- **Time** is the measure of the succession of events. Often, we are more interested in the change of physical quantities with respect to time, e.g. $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, instead of time variable itself.
- **Force** —a fixed vector— is the measure of the attempt to move a body.
- **Particle** is a body of which its dimension is negligible. The rotation effect is insignificant because it is just a point. Whether the body can be treated as the particle or not depends on the relative dimensions in the problem and how much detailed of the solution we are interested in.
- **Rigid body** is a body whose relative movement between its parts are negligible relative to the gross motion of the body. For example the motion of an ingot can be analyzed by assuming the object being rigid.
- **Nonrigid body** is a body whose relative movement between its parts are significant relative to the gross motion of the body. Knowledge of the mechanics of the deformable material must be used along with Dynamics in order to determine the absolute motion of the nonrigid bodies.

Let us consider some examples to see the difference of each term. If we have an object and consider the very small substance of the body. For differential element analysis of the body, the small substance can be treated as a particle. However, the substance must be handled as connecting objects had the molecular effects in the body are of concern. Or think of an airplane. Even of its huge size, the whole airplane may be modeled as a point in flight speed analysis along the route. But if the rotational motion, such as yawing or pitching, of the airplane body is important, its size does matter.

The next two examples are to show whether an object is considered rigid or nonrigid depends on how much detailed of the problem we would like to analyze. Truss can just be looked as a rigid body for the preliminary design of truss structure. But we must think of the truss elasticity if we were to choose the material for that truss. A stiff linkage of the robot may be considered a rigid body. However, the n-connecting linkages, treated as a whole, to form the robot arm is an example of nonrigid body. Note the body-fixed inertia of the nonrigid body is not constant.

1.2 Scalars and Vectors

In this section, we describe the scalar and vector quantities. Particularly, the vector term is explained more in length because it is fundamental to many dynamical variables.

- **Scalars** are quantities for which only the magnitude can describe completely. Time, volume, density, speed, energy, and mass are some examples.

- **Vectors** are quantities for which both the magnitude and the direction are needed to completely describe. Examples are displacement, velocity, acceleration, force, moment, and momentum. Vectors cand be classified into 3 types: free vector, sliding vector, and fixed vector.

- **Free Vector** is a vector whose action is not confined with a unique line in space. That is, only its *magnitude* and *direction* do matter. Some examples are the *displacement vector* of a pure translational rigid object, or the *couple vector* of a rigid body. Free vector is free to slide and translate as long as its direction and magnitude are maintained. In other words, its line of action and point of application do not matter.

- **Sliding Vector** is a vector whose *line of action* must be specified in addition to its magnitude and direction. External force or moment acting on the rigid body falls under this category. Therefore sliding vector has a freedom to *slide* along the fixed line of action.

- **Fixed Vector** is a vector whose magnitude, direction, line of action, and *point of application* are all important in the analysis. External force or moment acting onto the nonrigid body must be dealt with as the fixed vector due to the deformable effect of the object.

1.2.1 Representation of Vectors

There are many notations to represent a vector quantity, i.e. $\mathbf{v}, \bar{v}, \vec{v}$, or $\underline{v}$. If we would like to tell only the magnitude, $|\mathbf{v}|$ or v may be used. Keep in mind that the complete representation of a vector must be able to determine its magnitude, direction, line of action, and point of application. See Fig. 1.1.

1.2.2 Vector Manipulation

There are several ways which involve many procedures in adding two vector quantities, $\mathbf{v} = \mathbf{v_1} + \mathbf{v_2}$. See Fig. 1.2. First we have two arbitrary vectors, $\mathbf{v_1}$ and $\mathbf{v_2}$. By graphical approach, which is the clearest illustration, we use the *principle of transmissibility* to move each vector along its line of action so that their origin

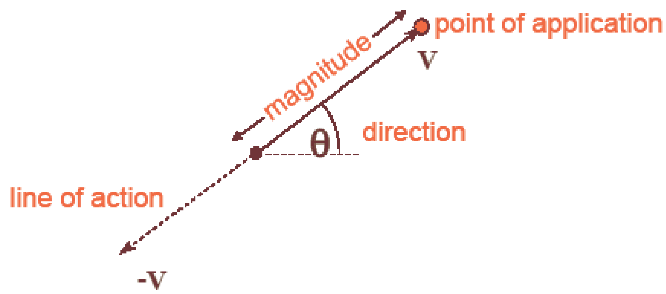


Figure 1.1: Magnitude, direction, line of action, and point of application of a vector ([1], pp. 5)

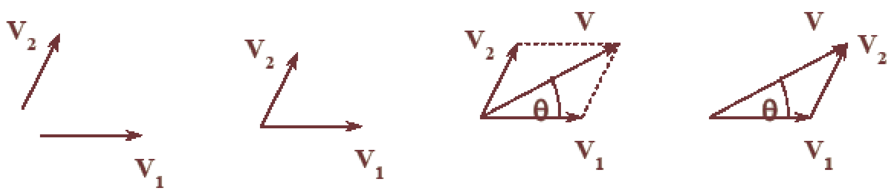


Figure 1.2: Parallelogram law and Head to tail for vector addition ([1], pp. 6)

points coincide. Then *parallelogram law* is applied to find the resultant vector, **v**. Vectors can also be added up using the *head to tail*, as done in the last figure. However, in contrary to the parallelogram method, the head to tail method does not guarantee correct line of action of the resultant vector. It may be obtained separately with the help of the *principle of moment*. Neither method gives the correct point of application. In summary, only the magnitude and direction of the resultant vector are ensured. If one choose the parallelogram law, the correct line of action can be obtained as well.

Some familiar algebraic laws also hold for the vector addition operation. They are

- **Commutative Law** The order of vectors in addition operation does not matter. $\mathbf{v_1} + \mathbf{v_2} = \mathbf{v_2} + \mathbf{v_1}$.
- **Associative Law** The order of vectors in addition operation does not matter. $(\mathbf{v_1} + \mathbf{v_2}) + \mathbf{v_3} = \mathbf{v_1} + (\mathbf{v_2} + \mathbf{v_3})$.
- **Vector Subtraction** The subtraction is the addition of the negative of the vector. $\mathbf{v_1} - \mathbf{v_2} = \mathbf{v_1} + (-\mathbf{v_2})$.

In the algebraic approach, cosine law and sine law are used in determining the magnitude and direction of the resultant vector from the addition of two vectors, as shown in Fig. 1.3. The resultant vector, drawn in red, is one side of the triangle

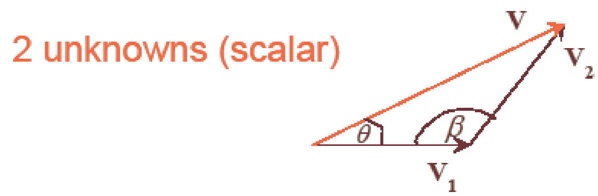


Figure 1.3: Determination of the magnitude and direction of the resultant vector

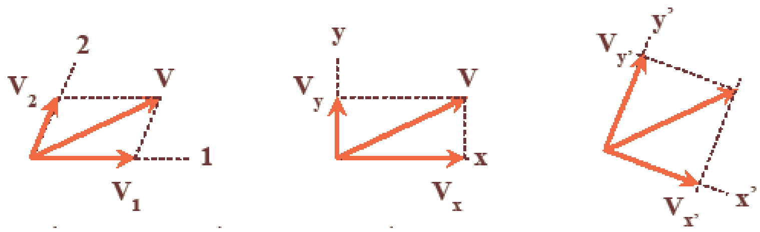


Figure 1.4: Vector components along different coordinate systems ([1], pp. 6)

forming from the other two vectors, $\mathbf{v_1}$ and $\mathbf{v_2}$. The magnitude, i.e. the length, of the resultant vector $\mathbf{v}$ is found by the cosine law:

$$v^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\beta \tag{1.1}$$

And the direction can be determined from the sine law and pruning the impossible solution:

$$\frac{v}{\sin\beta} = \frac{v_2}{\sin\theta} \tag{1.2}$$

1.2.3 Coordinate Systems

Coordinate systems are used to describe systematically the vectors. Different coordinate systems can be defined and used to solve the same problem because the vector quantities are invariant to the coordinate systems. However, some of them will be more appropriate to the problem at hand than others.

Usually we get used to the coordinate systems of which their coordinate axes are perpendicular. They are called rectangular coordinate systems. In some situation, non-rectangular coordinate system may be needed.

After we set up the coordinate system, the vector can be described by its *components* along the coordinate axis directions. As seen in Fig. 1.4, the same vector $\mathbf{v}$ can be described in many ways depending on the coordinate system used.

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_x + \mathbf{v}_y = \mathbf{v}_{x'} + \mathbf{v}_{y'}$$

Also, it can be seen that the vector components can be found by the use of *parallelogram law*. Vector components are the adjacent sides of the parallelogram. Therefore, cosine and sine laws can be used to determine the components.

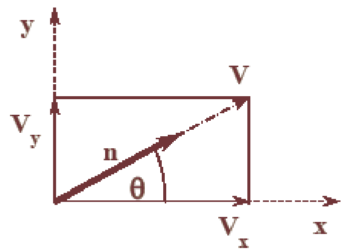


Figure 1.5: 2-D rectangular coordinate system

In case of rectangular coordinate system, vector components are determined simply by the dot product of the vector with unit vector along those axes. In other words, the components of the vector associated with the given *rectangular* coordinate system are the *orthogonal* projection of the vector onto the corresponding coordinate axes.

1.2.4 2-D Rectangular Coordinate System

We focus on the basic relationship of the vector and its components in 2-D rectangular coordinate system, shown in Fig 1.5. The components in this special case is the orthogonal projection of the vector onto the corresponding coordinate axes, which can be calculated from the dot product of the vector with the respective unit vector.

$$v_x = \mathbf{v} \cdot \mathbf{i} = v \cos\theta \tag{1.3}$$
$$v_y = \mathbf{v} \cdot \mathbf{j} = v \sin\theta \tag{1.4}$$

1.2.5 3-D Rectangular Coordinate System

Let us now focus on the basic relationship of the vector and its components in 3-D rectangular coordinate system, shown in Fig. 1.6. The components in this special case is the orthogonal projection of the vector onto the corresponding coordinate axes, which can be calculated from the dot product of the vector with the respective unit vector.

$$v_x = \mathbf{v} \cdot \mathbf{i} = v \cos\theta_x \tag{1.5}$$
$$v_y = \mathbf{v} \cdot \mathbf{j} = v \cos\theta_y \tag{1.6}$$
$$v_z = \mathbf{v} \cdot \mathbf{k} = v \cos\theta_z \tag{1.7}$$

$\cos\theta_x, \cos\theta_y, \text{and} \cos\theta_z$ are called the *direction cosine* of the vector since they give information of the vector direction. Because the coordinate system used is rectangular, it follows that

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1 \tag{1.8}$$

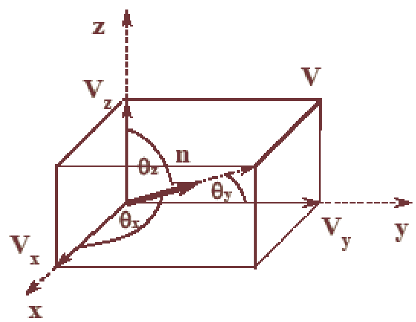


Figure 1.6: 3-D rectangular coordinate system

It can be concluded from the equation that the three angles are dependent. Only two of them are enough to specify the orientation of the vector.

1.3 Newton's Laws

In this section, we briefly mention the Newton's laws that describe the motion of the particle under low velocity. The **first law** states:

> "A *particle* remains at rest or continue to move in a straight line with a uniform velocity if there is no unbalanced force acting on it."

This statement can be formulated as

$$\Sigma \mathbf{F} = \mathbf{0} \Leftrightarrow \mathbf{a} = \mathbf{0}$$

Newton's **second law**, the most well-known of three, states:

> "The *absolute* acceleration of a *particle* is proportional to the resultant force acting on it and is in the direction of this resultant force."

This statement can be formulated as

$$\Sigma \mathbf{F} = m\mathbf{a} \tag{1.9}$$

where $\mathbf{a}$ = *absolute* acceleration of the particle.

Newton's **third law** states:

> "The forces of *action* and *reaction* between interacting bodies are equal in magnitude, opposite in direction, and collinear."

It can be nmemonically written as

$$\text{action force} = -\,(\text{reaction force})$$

This fact is used very often in drawing the free body diagram (FBD). See Fig. 1.7.

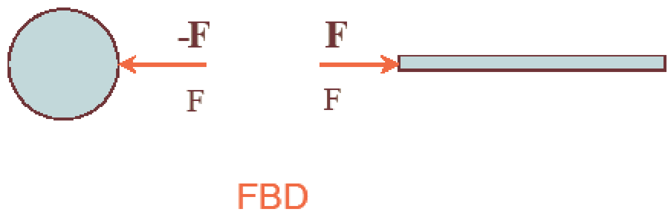


Figure 1.7: Newton's third law

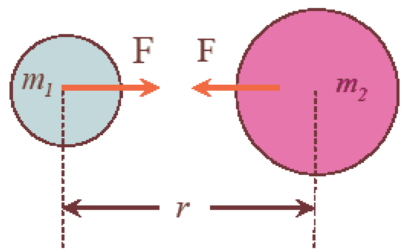


Figure 1.8: Gravitational force

1.4 Gravitational Law

Any two bodies have the attraction force governed by the gravitational law (Fig. 1.8):

$$F = G\frac{m_1 m_2}{r^2} \tag{1.10}$$

where

$$\begin{aligned} F &= \text{attraction force} \\ G &= \text{gravitational constant value} = 6.673 \times 10^{-11} \quad \text{m}^3/\left(\text{kg} \cdot \text{s}^2\right) \\ m &= \text{mass of the involving bodies} \\ r &= \text{distance between the bodies} \end{aligned}$$

Hence there is always the attraction force between the earth and the object. This gravitational force is called the *weight* of the body.

$$W = m\frac{Gm_e}{r^2} = mg \tag{1.11}$$

where

$$\begin{aligned} g &= \text{free falling acceleration observed on the } \textbf{moving} \text{ earth} \\ &= 9.81 \quad \text{m/s}^2 \end{aligned}$$

In practice, however, the gravitational acceleration can be considered the **absolute** acceleration for the engineering problem on earth.

Chapter 2

Force Systems

2.1 Overview of Forces

This chapter explain the force systems in 2-D and 3-D. The rectangular coordinate system is set up as a mean to describe the force system. Force and their consequences, moment and couple, can then be described using the defined coordinate system. Finally, resultant force system will be determined as the simplest representation of the complex system of forces, moments, and couples.

In this section, we roughly explain the concepts and basic terminologies involving with force.

- **Force** is the measure of the attempt to *move a body*. It is a *fixed vector*. For the rigid body problems, or if only the external effects of the external force onto the objects are of interested, that force can be treated as a *sliding vector*. Hence the problem can make use of the *principle of transmissibility*. Specification of the magnitude, direction, and line of action can completely describe the force vector.

  Figure 2.1 illustrates the use of the principle of transmissibility. If the body is rigid (enough), the force pushing at point A will generate the same resulting motion as the force pulling the object at point B.

- **Contact force vs. Body force** *Contact force* occurs from the contact between the bodies, which is very common since the phenomena can be easily observed. However, there is force between the bodies even though they are not in contact. It is the attracting force, or *body force*, trying to pull the bodies together, of which its magnitude is governed by the gravitational law.

- **Concentrated force vs. Distributed force** Most of the forces are naturally the *distributed force*, which means the force acts over some area or surface such as the force developed between the tire and the road. *Concentrated force* is the ideal situation that the surface or volume where the force acts on is negligible. This simplifies the situation and makes possible the preliminary analysis of the problem. In some circumstances, where the body can be assumed rigid, the distributed force can be reduced to a single resultant force. In other words, we find the equivalent concentrated force to the original distributed force system.

- **Force measurement** Force sensor is used to measure the force. Its principle is to infer the force from the deformation of an elastic element. Calibration of the force sensor to the known load is necessary.

- **Action vs. Reaction force** They are pair of forces expressing the interaction between bodies. They will be revealed when isolate the surrounding objects from the system of interest. Free body diagram (FBD) is used to help indentify the action and reaction forces.

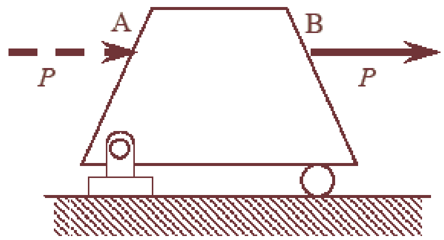


Figure 2.1: For rigid body, the force can be moved along the line of action and the effect is unchanged

- **Combining force** We use the parallelogram law or the principle of moment, in addition to the conventional vector addition, to combine system of forces. The line of action of the combined force will be corrected.

- **Force components** Determining the force components along the specified coordinate system is the reverse procedure of combining the forces. Again, the parallelogram law or the principle of moment, including the principle of transmissibility are used to obtain the correct force components.

- **Orthogonal projection** is the perpendicular projection along the specified direction. It is calculated by the dot product of the vector and the unit vector in that direction. The components of a vector, however, are usually not the same as the orthogonal projection onto the same coordinate system. Exception is the orthogonal (rectangular) coordinate system. Figure 2.2 depicts the distiction. With the parallelogram law, sum of the components must equal to the original vector.

- **Addition of parallel forces** by the graphical method has the problem that the line of action of the resulting force cannot be determined as usual by the principle of transmissibility and the parallelogram law because the line of actions do not intersect. The trick is to make them intersect by adding an arbitrary force vector to one of the original force, and then subtracting the opposite from the other one of the original force. The new pair of forces is now unparalleled and can be added up to get the resultant force with the location of the line of action. Figure 2.3 shows the steps of the parallel forces addition.

2.2 2-D Rectangular Coordinate Systems

Figure 2.4 shows the force vector and its rectangular components. If a 2-D rectangular coordinate system has been specified, a planar force vector, **F**, can be written as the addition of its component vectors along the coordinate axes.

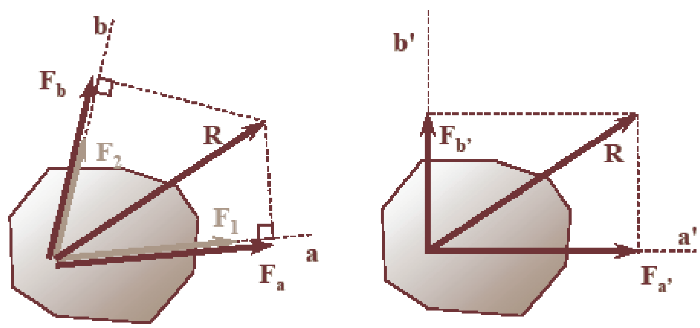


Figure 2.2: Components vs. orthogonal projection of a vector onto the same coordinate system

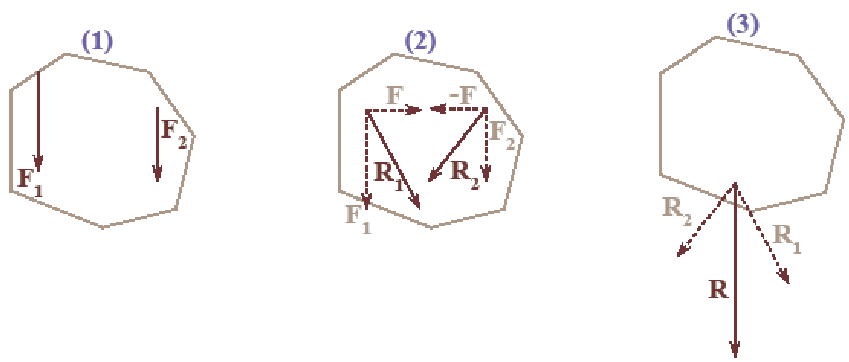


Figure 2.3: Parallel force addition

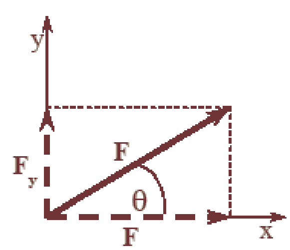


Figure 2.4: Planar force vector and its rectangular components

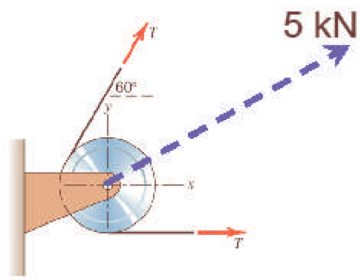


Figure 2.5: Example 2.1 ([2], pp. 29)

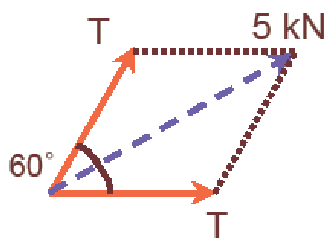


Figure 2.6: Solution to example 2.1

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y = F_x\mathbf{i} + F_y\mathbf{j} \tag{2.1}$$

The components are the orthogonal projection of the vector onto the respective axes which are determined by the dot product of the vector and the unit vector along the axes.

$$\begin{aligned} F_x &= \mathbf{F}\cdot\mathbf{i} = F\cos\theta \\ F_y &= \mathbf{F}\cdot\mathbf{j} = F\sin\theta \end{aligned} \tag{2.2}$$

The magnitude and direction of the force vector **F** follow immediately as

$$F = \sqrt{F_x^2 + F_y^2} \tag{2.3}$$

$$\theta = \text{arctan2}\left(F_y,\ F_x\right) \tag{2.4}$$

Example 2.1 ([2], Prob. 2/9) If the two equal tension **T** in the pulley cable together produce a force of 5 kN on the pulley bearing, calculate **T**.

Solution: Use the parallelogram law and the cosine law to determine the non-orthogonal components. From the force vector addition and by the cosine law,

$$\begin{aligned} 5^2 &= T^2 + T^2 + 2T \times T\cos 60^\circ \\ T &= 2.89 \ \ \text{kN} \end{aligned}$$

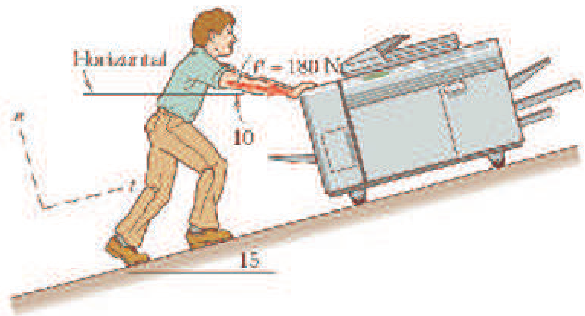


Figure 2.7: Example 2.2 ([2], pp. 30)

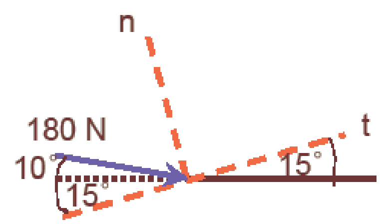


Figure 2.8: Solution to example 2.2

Example 2.2 ([2], Prob. 2/11) While steadily pushing the machine up an incline, a person exerts a 180 N force **P** as shown. Determine the components of **P** which are parallel and perpendicular to the incline.

Solution: Draw the coordinate axes and the force vector. Carefully indicate the angles. Then project the force to the respective axes.

$$\begin{aligned} P_t &= 180\cos(10+15) &= 163.1\ \text{N} \\ P_n &= -180\sin(10+15) &= -76.1\ \text{N} \end{aligned}$$

Example 2.3 ([2], Prob. 2/19) Determine the resultant **R** of the two forces applied to the bracket. Write **R** in terms of unit vectors along the x- and y- axes shown.

Solution: Draw the coordinate axes and the force vectors. Carefully indicate the angles. After that, project the forces to the respective axes. Then algebraically add the components that are on the same axis to obtain the answer. The problem is extended to determine the components along the non-orthogonal coordinate system.

Force components along the x-y coordinate system are

$$\begin{aligned} R_x &= 200\cos(15+20) - 150\sin(10+20) &= 88.8\ \text{N} \\ R_y &= 200\sin(15+20) + 150\cos(10+20) &= 244.6\ \text{N} \end{aligned}$$

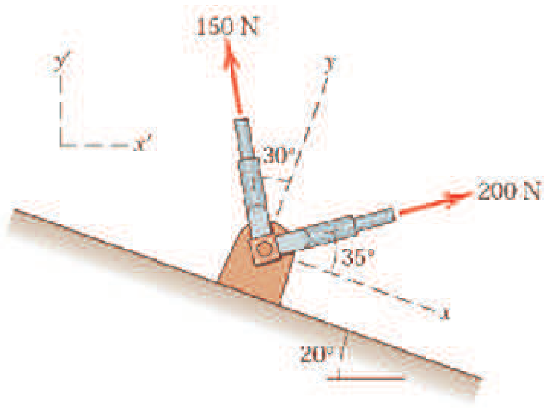


Figure 2.9: Example 2.3 ([2], pp. 32)

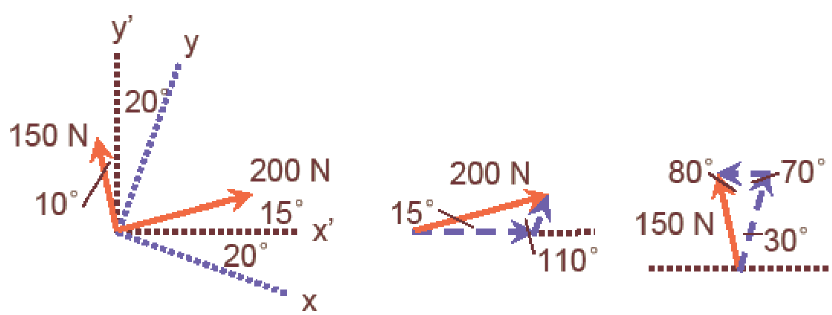


Figure 2.10: Solution to example 2.3

Force components along the x'-y' coordinate system are

$$\begin{aligned} R_{x'} &= 200\cos 15 - 150\sin 10 &= 167.1 \quad \text{N} \\ R_{y'} &= 200\sin 15 + 150\cos 10 &= 199.5 \quad \text{N} \end{aligned}$$

Hence the resultant **R** is

$$\mathbf{R} = 88.8\mathbf{i} + 244.6\mathbf{j} \quad \text{N} = 167.1\mathbf{i}' + 199.5\mathbf{j}' \quad \text{N}$$

Force components along the non-orthogonal x'-y coordinate system are determined by law of sine and cosine:

$$200 \quad \text{N} \Longrightarrow 174.34\mathbf{i}' + 55.1\mathbf{j} \quad \text{N} \quad \text{and} \quad 150 \quad \text{N} \Longrightarrow -79.8\mathbf{i}' + 157.2\mathbf{j} \quad \text{N}$$

$$\mathbf{R} = (174.34 - 79.8)\,\mathbf{i}' + (55.1 + 157.2)\,\mathbf{j} = 94.54\mathbf{i}' + 212.3\mathbf{j} \quad \text{N}$$

Example 2.4 ([1], Prob. 2/20) It is desired to remove the spike from the timber by applying force along its horizontal axis. An obstruction A prevents direct access, so that two forces, one 1.6 kN and the other **P**, are applied by cables as

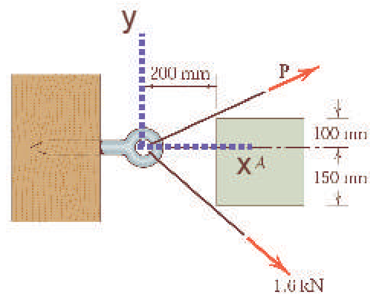


Figure 2.11: Example and solution of 2.4 ([1], pp. 36)

shown. Compute the magnitude of **P** necessary to ensure axial tension **P** along the spike. Also find **T**.

Solution: To remove the spike, the efforted force must point in the direction along the spike axis. This means there is no net force in the perpendicular direction. **P** and 1.6 kN must add up to give the resultant force in the horizontal direction.

Because there is no net force in the y- direction, the summation of force is

$$R_y = P \sin\left(\text{atan}\left(\frac{100}{200}\right)\right) - 1.6 \sin\left(\text{atan}\left(\frac{150}{200}\right)\right) = 0$$

Therefore,

$$P = 2.15 \quad \text{kN}$$

The axial tension is the total force along the x- direction.

$$T = R_x = P \cos\left(\text{atan}\left(\frac{100}{200}\right)\right) + 1.6 \cos\left(\text{atan}\left(\frac{150}{200}\right)\right) = 3.20 \quad \text{kN}$$

Example 2.5 ([1], Prob. 2/26) As it inserts the small cylindrical part into a close fitting circular hole, the robot arm exerts a 90 N force **P** on the part parallel to the axis of the hole as shown. Determine the components of the force which the part exerts on the robot along axes *(a)* parallel and perpendicular to the arm AB, and *(b)* parallel and perpendicular to the arm BC.

Solution: The indicated force **P** is the force done by the robot on the cylindrical part. Therefore the force exerted by the part on the robot is $-\mathbf{P}$. We set up the coordinate frame $n_1 t_1$ and $n_2 t_2$ where their axes are perpendicular and parallel to the link AB and BC, respectively. Then $-\mathbf{P}$ is projected onto these rectangular coordinate system.

Force done by the part on the robot written in $n_1 t_1$ frame is

$$-\mathbf{P} = -90 \cos 45 \mathbf{n}_1 + 90 \sin 45 \mathbf{t}_1 = -63.6 \mathbf{n}_1 + 63.6 \mathbf{t}_1 \quad \text{N}$$

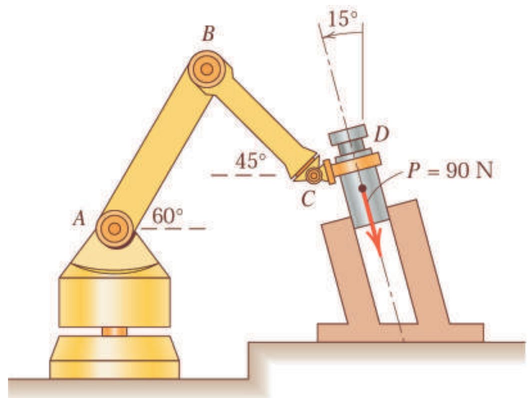


Figure 2.12: Example 2.5 ([1], pp. 37)

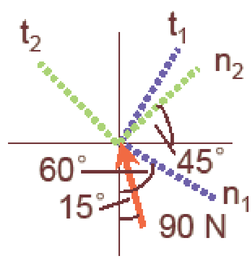


Figure 2.13: Solution to example 2.5

Force done by the part on the robot written in $n_2 t_2$ frame is

$$-\mathbf{P} = -90\cos 60\mathbf{n}_2 + 90\sin 60\mathbf{t}_2 = 45\mathbf{n}_2 + 77.9\mathbf{t}_2 \quad \text{N}$$

2.3 2-D Force, Moment, and Couple

Moment is the measure of the attempt to *rotate* a body, which is usually induced by force. The moment is always associated with a specified point, meaning that we must specify the point in determining the moment *about that point*. In 2-D problems, the moment vector's direction is always perpendicular to the plane established by the point and the line of action of the force. In this course, the moment can be treated as a sliding vector so the problem can make use of the *principle of transmissibility*.

Moment of the force $\mathbf{F}$ about point A, fig. 2.14, is

$$\mathbf{M}_A = \mathbf{r} \times \mathbf{F} \tag{2.5}$$

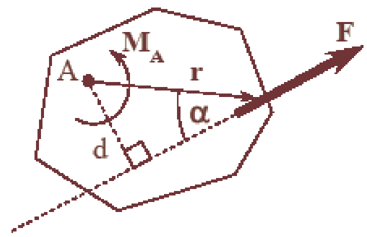


Figure 2.14: Moment $\mathbf{M}_A$ of $\mathbf{F}$ about point A

where $\mathbf{r}$ is the position vector from A to any point along the line of action of $\mathbf{F}$. If only the magnitude is considered, the formula can be written as

$$M_A = F \cdot r\,(\sin\alpha) = F \cdot d \tag{2.6}$$

where d is the perpendicular distance from the line of action to point A.

In 2-D problems, the moment vector always points perpendicular to the plane. Therefore, it can also be specified by the *magnitude* and the *sense of rotation* about the point. It does not matter in which direction we will assign positive value. However, the sign consistency throughout the problem must be kept.

Varignon's Theorem, or the *Principle of Moment*, states that

> "The moment of a force about any point is equal to the sum of the moments of the components of the force about the same point"

Figure 2.15 illustrates the principle. Mathematically, if $\mathbf{F} = \mathbf{P} + \mathbf{Q}$,

$$(\mathbf{M}_A)_\mathbf{F} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{P} + \mathbf{Q}) = (\mathbf{M}_A)_\mathbf{P} + (\mathbf{M}_A)_\mathbf{Q} \tag{2.7}$$

This theorem is helpful in determining the moment of the force from its components. In many cases, moments of some components may be trivial to calculate.

Couple is the measure of the attempt to *purely rotate* a body. It can be produced by two equal, opposite, and non-collinear forces. Calculation of the couple is depicted in fig. 2.16.

$$\mathbf{M} = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = \mathbf{r} \times \mathbf{F} \tag{2.8}$$

$$M = F \cdot r\,(\sin\alpha) = Fd \tag{2.9}$$

The couple vector's direction is *perpendicular to the plane* established by those two lines of action of the forces. It is a *free vector* and so *no moment center*. Only the magnitude and direction are enough to describe the couple.

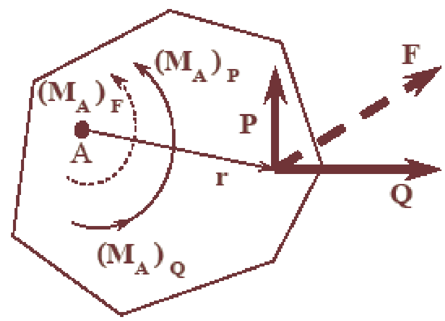


Figure 2.15: Principle of moment

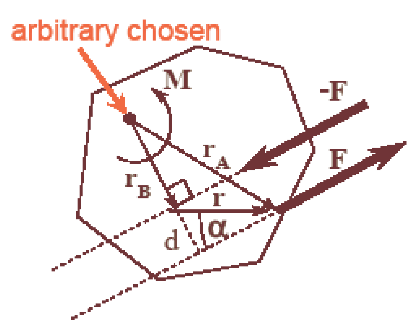


Figure 2.16: Two dimensional couple

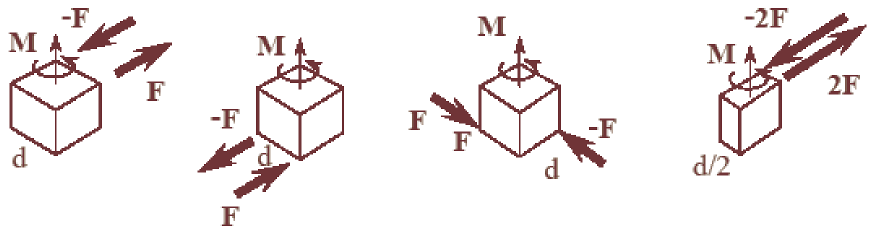


Figure 2.17: Couple generated by equal and opposite pair of forces

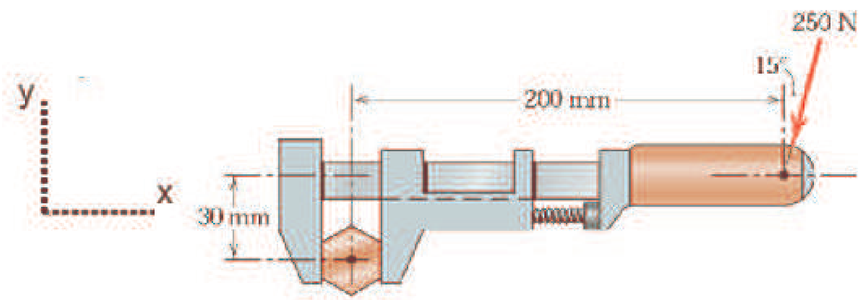


Figure 2.18: Example and solution of 2.6 ([1], pp. 44)

For rigid body, several pairs of equal and opposite forces can give the same couple. Therefore it is unique to calculate the couple from a given pair of forces *but* it is non-unique to determine the pair of forces which will produce that value of couple. See fig. 2.17. This property can be applied in determining the effect of the couple from the equivalent pair of forces. Effect from some specific pair of forces may be trivial to calculate.

Example 2.6 ([1], Prob. 2/36) Calculate the moment of the 250 N force on the handle of the monkey wrench about the center of the bolt.

Solution: There are many ways in determining the answer. The most suitable way, however, is to factor the force into components where the moments can be found easily. Varignon's theorem is then used in calculating the desired moment from the sum of moments of their components.

Project 250 N force into x-y coordinate frame and sum the moments induced by these components about the center of the bolt.

$$M_O = -250\cos 15 \times 0.2 + 250\sin 15 \times 0.03 = 46.4 \quad \text{Nm} \qquad \text{CW}$$

Example 2.7 Calculate the moment of the 240 N force on the handle of the prong about the instantaneous supporting point O.

Solution: Here we illustrate the moment calculation by the vector approach. **r**

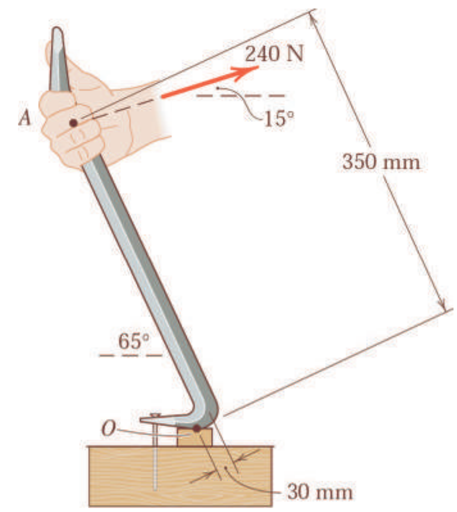


Figure 2.19: Example and solution of 2.7

and **F** are described as vectors according to the coordinate system. The moment can then be calculated directly as $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.

$$\begin{aligned} \mathbf{r} &= 0.03\mathbf{i} + 0.35\mathbf{j} \quad \text{m} \\ \mathbf{F} &= 240\cos 10\mathbf{i} - 240\sin 10\mathbf{j} \quad \text{N} \\ \mathbf{M}_O &= \mathbf{r} \times \mathbf{F} = -84.0\mathbf{k} \quad \text{Nm} \end{aligned}$$

Example 2.8 ([1], Prob. 2/46) The force exerted by the plunger of cylinder AB on the door is 40 N directed along the line AB, and this force tends to keep the door closed. Compute the moment of this force about the hinge O. What force $\mathbf{F}_C$ normal to the plane of the door must the door stop at C exert on the door so that the combined moment about O of the two forces is zero?

Solution: Hydraulic force is decomposed into horizontal and vertical direction, which is used in determining its moment about point O. Force $\mathbf{F}_C$ at the stopper to balance the moment can be easily calculated.

The angle which 40 N force made to the horizontal direction is

$$\theta = \text{atan}\,(100/400) = 0.245 \quad \text{rad}$$

Hence the moment of the hydraulic force about point O is

$$M_O = -40\cos\theta \times 0.075 - 40\sin\theta \times 0.425 = 7.03 \quad \text{Nm} \qquad \text{CW}$$

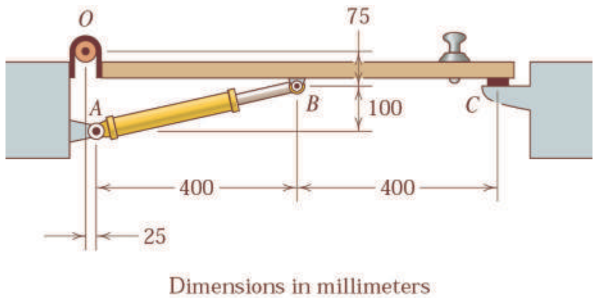


Figure 2.20: Example 2.8 ([1], pp. 47)

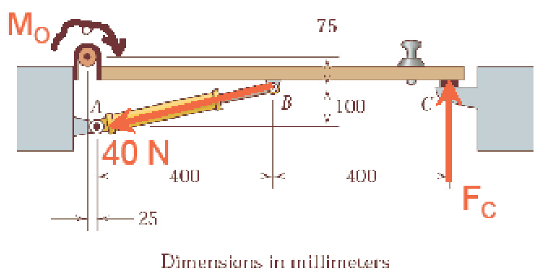


Figure 2.21: Solution to example 2.8

The force F_C must balance this moment. Therefore

$$F_C = M_O/0.825 = 8.53 \quad \text{N}$$

Example 2.9 ([1], Prob. 2/52) While inserting a cylindrical part into the circular hole, the robot exerts the 90 N force on the part as shown. Determine the moment about point A, B, and C of the force which the part exerts on the robot.

Solution: The coordinate system is first set up. $\mathbf{M}_C$ is determined readily. $\mathbf{M}_B$ and $\mathbf{M}_A$ are then determined by adding the extra moment to $\mathbf{M}_C$ caused by the moment of $-\mathbf{P}$ at C about B and A respectively. Note the problem asks the moment of the force which the part exerts on the robot, that is the moment of $-\mathbf{P}$.

Force which the part exerts on the robot is

$$\mathbf{F} = -\mathbf{P} = -90\sin 15\mathbf{i} + 90\cos 15\mathbf{j} = -23.29\mathbf{i} + 86.93\mathbf{j} \quad \text{N}$$

Moment about point C is

$$M_C = 90 \times 0.15 = 13.5 \quad \text{Nm} \qquad \text{CCW}$$

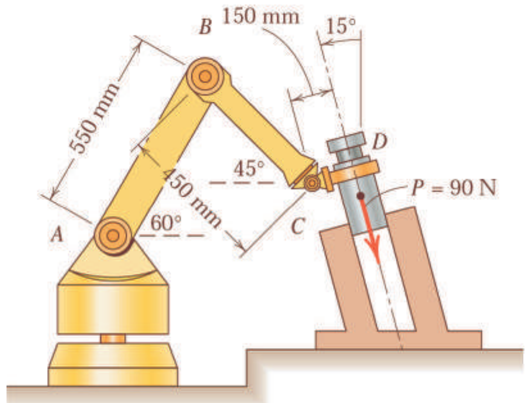


Figure 2.22: Example 2.9 ([1], pp. 48)

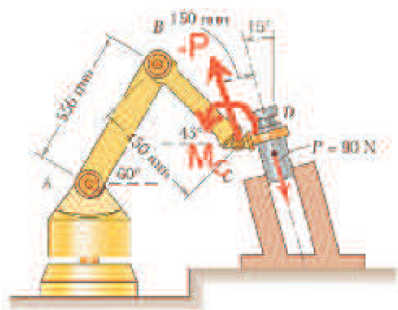


Figure 2.23: Solution to example 2.9

From fig. 2.23, the position vector $\mathbf{r}_{AC}$ written in x-y coordinate system is

$$\mathbf{r}_{AC} = (0.55\cos 60 + 0.45\cos 45)\,\mathbf{i} + (0.55\sin 60 - 0.45\sin 45)\,\mathbf{j} = 0.593\mathbf{i} + 0.158\mathbf{j} \ \text{ m}$$

Hence the moment of $-\mathbf{P}$ at C about A is

$$\mathbf{M}_{-\mathbf{P} \text{ at } C \text{ about } A} = \mathbf{r}_{AC} \times \mathbf{F} = 55.23 \ \text{ Nm} \quad \text{ CCW}$$

Therefore moment of the force which the part exerts on the robot about A is

$$\mathbf{M}_A = \mathbf{M}_C + \mathbf{M}_{-\mathbf{P} \text{ at } C \text{ about } A} = 68.7 \ \text{ Nm} \quad \text{ CCW}$$

Example 2.10 ([2], Prob. 2/55) As part of a test, the two aircraft engines are revved up and the propeller pitches are adjusted so as to result in the fore and aft thrusts shown. What force **F** must be exerted by the ground on each of the main braked wheels at A and B to counteract the turning effect of the two propeller

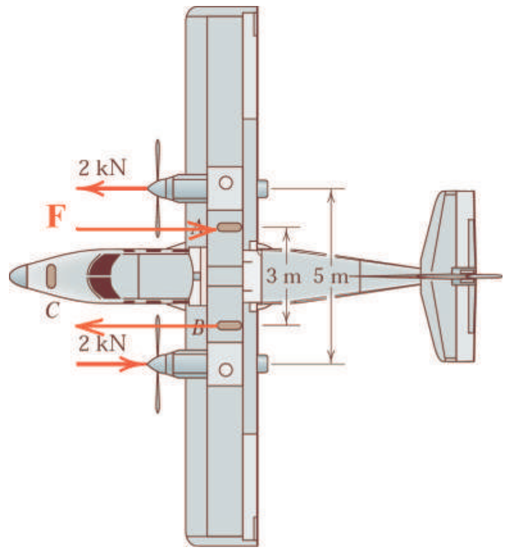


Figure 2.24: Example and solution of 2.10 ([2], pp. 48)

thrusts? Neglect any effect of the nose wheel C, which is turned 90° and unbraked.

Solution: Calculate the couple generated by the thrust forces and equate it to the one produced by the braked forces. The resultant couple is zero, that is,

$$M_C = 2 \times 5 - F \times 3 = 0$$

$$F = 3.33 \;\; \text{kN}$$

Example 2.11 ([2], Prob. 2/59) A lug wrench is used to tighten a square-head bolt. If 250 N forces are applied to the wrench as shown, determine the magnitude F of the equal forces exerted on the four contact points on the 25 mm bolt head so that their external effect on the bolt is equivalent to that of the two 250 N forces. Assume that the forces are perpendicular to the flats of the bolt head.

Solution: Calculate the couple generated by the two 250 N forces. Four forces exerted at the sides of the square-head bolt are such that they give the negative value to balance the applied couple. The equivalent couple system at the bolt head is shown in fig. 2.26.

$$\begin{aligned} 250 \times 0.7 &= 2\,(F \times 0.025) \\ F &= 3500 \;\; \text{N} \end{aligned}$$

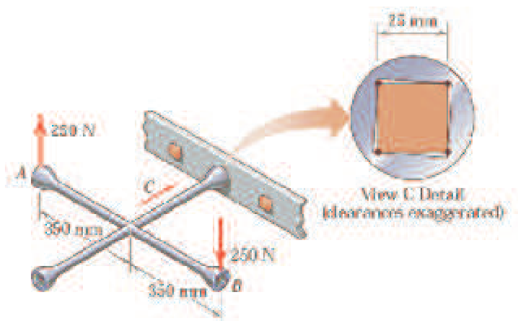


Figure 2.25: Example 2.11 ([2], pp. 49)

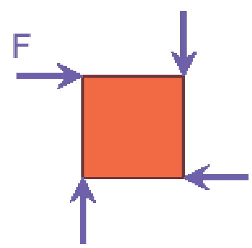


Figure 2.26: Solution to example 2.11

2.4 2-D Resultants

Force has the direct effect of *pushing* or *pulling* the body in the direction of the force. It has the indirect effect, however, of *rotating* the body about any axis except the intersection line to the line of force. Sometimes it is useful to separate the pushing/pulling and the rotating effect while maintaining the resultant force and moment. See fig. 2.27.

Force **F**, alone, in the left figure has the effect of moving point B along the vector direction. It also tries to rotate the object about point B. However, force **F** in the right figure only has the effect of moving point B along the vector direction. The rotation about point B is accounted by the couple $M = Fd$, which can be imagined by adding and subtracting **F** and $-$**F** at point B. Force system

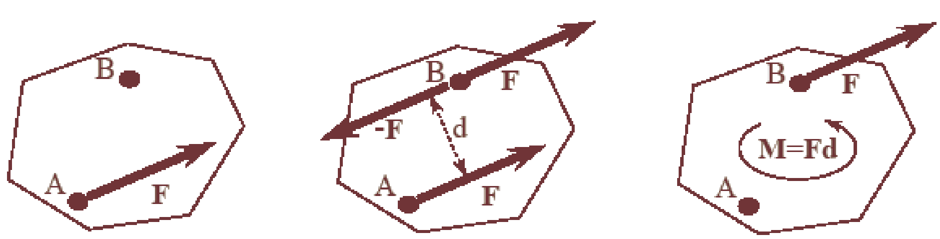


Figure 2.27: Force-Couple system

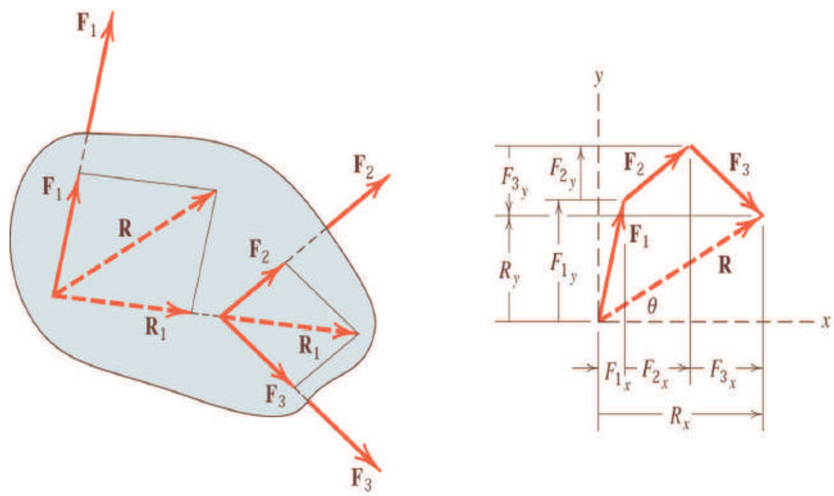


Figure 2.28: Force polygon method ([1], pp. 58)

of three figures are equivalent in the sense that they make the rigid body moves in the same manner. To repeat, force **F** acting at point A is equivalent to the force **F** acting at point B plus the compensated couple M.

Resultant is the simplest force combination which can replace the original system of forces, moments, and couples without altering the external effect of the system on the rigid body. Several tools are used in determining the resultant.

Force polygon method, or head to tail of the force vectors, may be used in determining the resultant force. However, only the magnitude and direction are ensured. That is, the *line of action* may be incorrect! Figure 2.28 illustrates the method in determining the resultant of three forces, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$.

For the specified rectangular coordinate system,

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \sum \mathbf{F}$$

$$R_x = \sum F_x, \quad R_y = \sum F_y, \quad R = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2} \tag{2.10}$$

$$\theta = \text{atan2}\left(R_y,\ R_x\right) \tag{2.11}$$

Principle of transmissibility and **Parallelogram law** can be used in determining the resultant force. This method gives the correct line of action. Therefore it is the quick and easy way to visualize the resultant but low

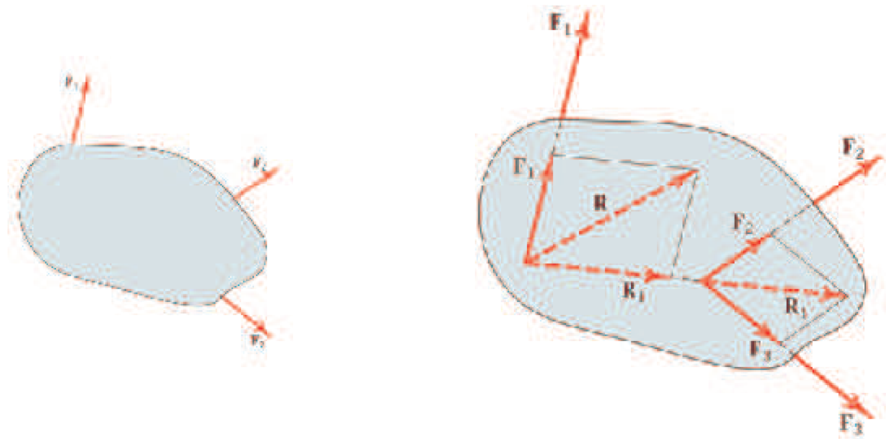


Figure 2.29: Determination of resultant force by the principle of transmissibility ([1], pp. 58)

accuracy. The magnitude, direction, and *line of action* obtained by this method are correct. Figure 2.29 shows an example of determining the resultant force by the principle of transmissiblity and the parallelogram law.

Force-Couple equivalent method is the algebraic approach in determining the resultant. Compensated couple happens to counter the effect of moving the force to a new location. This method also gives the correct magnitude, direction, and *line of action*. Following is the guideline in determining the resultant force.

1. Specify a convenient reference point O.
2. Move all forces so the new lines of action pass through point O. The new lines are parallel to the original ones. By the force-couple equivalent method, the couples are generated to preserve the motion effect of the original force system.
3. Sum forces and couples to $\mathbf{R}$ and $\mathbf{M}_O$.
4. Locate the correct line of action of the resultant force $\mathbf{R}$. This is done by applying the inverse of force-couple equivalent method. Force $\mathbf{R}$ must be moved to a new location so that the generated couple cancels the couple $\mathbf{M}_O$. Only the force is left and hence is the resultant force.

The related equations used in this method are

$$\mathbf{R} = \sum_i \mathbf{F}_i \tag{2.12}$$

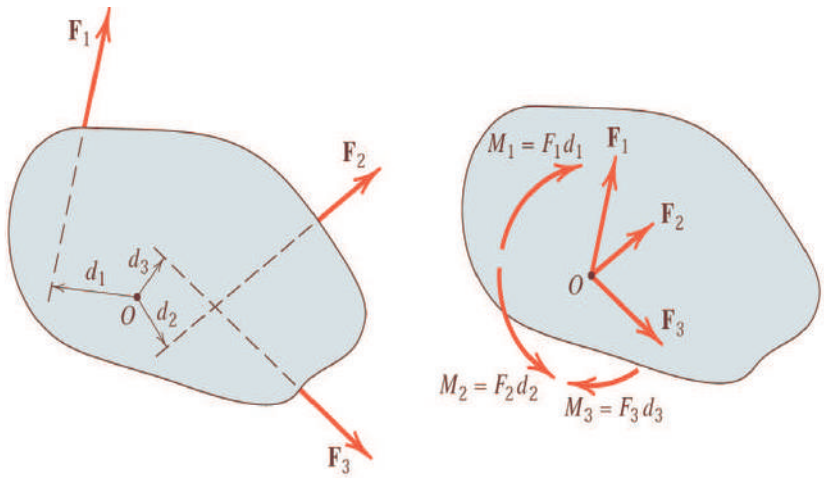


Figure 2.30: Step 1 and 2 of the force-couple equivalent method ([1], pp. 59)

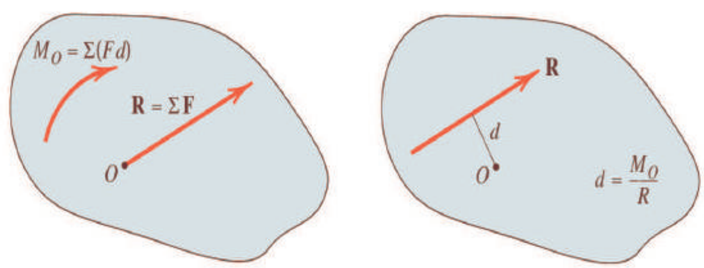


Figure 2.31: Step 3 and 4 of the force-couple equivalent method ([1], pp. 59)

to determine the magnitude and direction of the resultant, and

$$M_O = \sum_i M_i = \sum_i (F_i d_i) = Rd \tag{2.13}$$

which is the principle of moment to determine the line of action.

Example 2.12 ([2], Prob. 2/56) In the design of the lifting hook the action of the applied force **F** at the critical section of the hook is a direct pull at B and a couple. If the magnitude of the couple is 4000 Nm, determine the magnitude of **F**.

Solution: The moment induced by the resultant force must be equal to the 4000 Nm couple. Hence the resultant force magnitude can be determined.

$$F \times 0.1 = 4000, \quad F = 40 \ \text{kN}$$

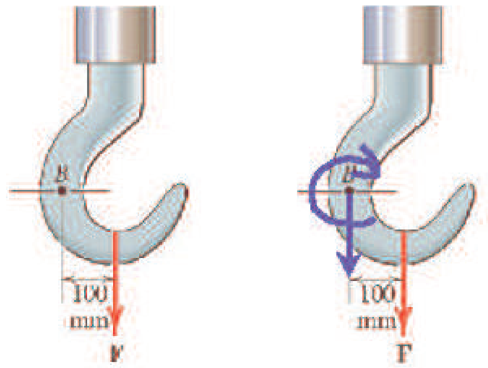


Figure 2.32: Example and solution of 2.12 ([2], pp. 49)

Example 2.13 ([2], Prob. 2/61) Calculate the moment of the 1200 N force about pin A of the bracket. Begin by replacing the 1200 N force by a force-couple system at point C. Calculate the moment of the 1200 N force about the pin at B.

Solution: Force-couple equivalent system sometimes makes the moment calculation intuitive. Here, we first determine the force-couple equivalence of the tension at C. Then we calculate the moment of the new equivalent system about point A and B.

Moment of 1200 N force about C is

$$M_C = 1200 \times 0.2 = 240 \text{ Nm CCW}$$

Therefore the moment about point A and B are

$$M_A = M_C + 1200 \times \frac{1}{\sqrt{5}} \times 0.6 = 562 \text{ Nm CCW}$$

$$M_B = M_A + 1200 \times \frac{2}{\sqrt{5}} \times 0.5 = 1099 \text{ Nm CCW}$$

Example 2.14 ([1], Prob. 2/70) The combined drive wheels of a front-wheel-drive automobile are acted on by a 7000 N normal reaction force and a friction force **F**, both of which are exerted by the road surface. If it is known that the resultant of these two forces makes a 15° angle with the vertical, determine the equivalent force-couple system at the car mass center G. Treat this as a 2D problem.

Solution: First, determine the resultant force of the normal and friction forces. Since the resultant force makes a 15° angle with the vertical,

$$R \cos 15 = 7000, \quad R = 7246.9 \text{ N}$$

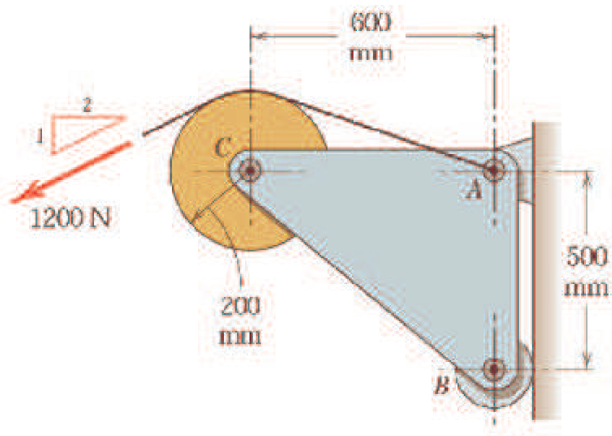


Figure 2.33: Example and solution of 2.13 ([2], pp. 50)

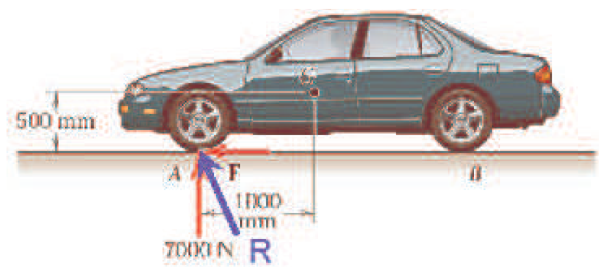


Figure 2.34: Example and solution of 2.14 ([1], pp. 56)

Then move the resultant force acting at the front wheel to the car mass center G. Consequently, the couple must be provided to compensate for the induced moment of the original force system.

$$M_G = 7000 \times 1 + 7246.9 \sin 15 \times 0.5 = 7937.8 \text{ Nm CW}$$

Example 2.15 ([2], Prob. 2/75) Determine and locate the resultant **R** of the two forces and one couple acting on the I-beam.

Solution: We first select an arbitrary point O to which all forces and couples will be moved to. Then the equivalent force-couple at point O is determined.

$$R = 8 - 5 = 3 \text{ kN downward}$$

$$M_O = 25 - 5 \times 2 - 8 \times 2 = 1 \text{ kNm CW}$$

Finally, the resultant force is found by locating the correct line of action. This step is essentially done by applying the principle of moment.

$$3d = 1, \quad d = 1/3 \text{ m} \ \& \ x = 4\frac{1}{3} \text{ m}$$

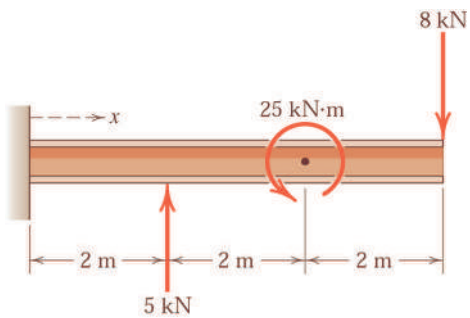


Figure 2.35: Example and solution of 2.15 ([2], pp. 56)

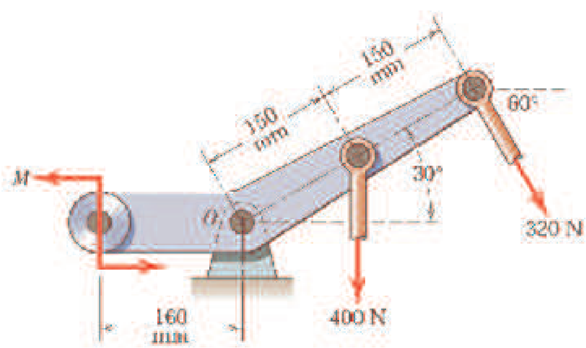


Figure 2.36: Example and solution of 2.16 ([1], pp. 62)

Example 2.16 ([1], Prob. 2/83) If the resultant of the two forces and couple **M** passes through point O, determine **M**.

Solution: The key to this problem is that the resultant passes through point O means there is no moment at point O. By summing all moment from the two forces and one unknown couple **M** at point O to zero, we can find the solution.

$$M_O = M - 400 \times 0.15 \cos 30 - 320 \times 0.3 = 0$$

$$M = 148 \quad \text{Nm} \quad \text{CCW}$$

Example 2.17 ([1], Prob. 2/88) The directions of the two thrust vectors of an experimental aircraft can be independently changed from the conventional forward direction within limits. For the thrust configuration shown, determine the equivalent force-couple system at point O. Then replace this force-couple system by a single force and specify the point on the x-axis through which the line of action of this resultant passes.

Figure 2.37: Example 2.17 ([1], pp. 63)

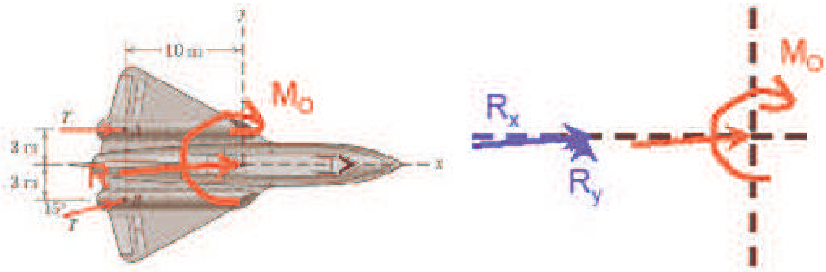


Figure 2.38: Solution to example 2.17

Solution: The force-couple equivalent system at point O is first determined as followed.

$$\mathbf{R} = (T + T\cos 15)\,\mathbf{i} + (T\sin 15)\,\mathbf{j} = 1.966T\mathbf{i} + 0.259T\mathbf{j} \quad \text{N}$$

$$M_O = T\cos 15 \times 3 - T \times 3 - T\sin 15 \times 10 = 2.69T \quad \text{Nm} \quad \text{CW}$$

Then the resultant force is the force located at which its moment about point O is equal to the couple. Since $\mathbf{R}_x$ of the new force system does not contribute moment about O, only $\mathbf{R}_y$ can be used in the calculation. The moment equation is used to solve for the position of the point through which the line of action of the resultant passes.

$$0.259T \times x = -2.69T, \quad x = -10.4 \quad \text{m}$$

Example 2.18 ([1], Prob. 2/93) Two integral pulleys are subjected to the belt tensions shown. If the resultant $\mathbf{R}$ of these forces passes through the center O, determine $\mathbf{T}$ and the magnitude of $\mathbf{R}$ and the CCW angle θ it makes with the x-axis.

Solution: The key to this problem is that the resultant passes through

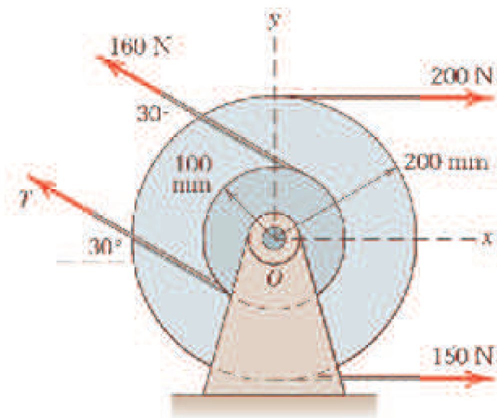


Figure 2.39: Example and solution of 2.18 ([1], pp. 64)

point O means there is no moment at point O. By summing all moments at point O from the tension forces to zero, $\mathbf{T}$ can be determined.

$$(160 - T) \times 100 + (150 - 200) \times 200 = 0, \quad T = 60 \quad \text{N}$$

$\mathbf{R}$ is obtained by summing the tension forces altogether.

$$\mathbf{R} = (200 + 150 - 160\cos 30 - 60\cos 30)\,\mathbf{i} + (160\sin 30 + 60\sin 30)\,\mathbf{j}$$

$$\mathbf{R} = 159.5\mathbf{i} + 110\mathbf{j} \quad \text{N}, \qquad R = 193.7 \quad \text{N} \quad \theta = 34.6^\circ$$

Example 2.19 ([1], Prob. 2/97) A rear-wheel-drive car is stuck in the snow between other park cars as shown. In an attempt to free the car, three students exert forces on the car at point A, B, and C while the driver's actions result in a forward thrust of 200 N acting parallel to the plane of rotation of each rear wheel. Treating the problem as 2D, determine the equivalent force-couple system at the car center of mass G and locate the position x of the point on the car centerline through which the resultant passes. Neglect all forces not shown.

Solution: The resultant force is simply the addition of all forces acting on the car. Using the given x-y coordinate system,

$$\mathbf{R} = (200 + 400 + 200 + 250\sin 30)\,\mathbf{i} + (250\cos 30 + 350)\,\mathbf{j} = 925\mathbf{i} + 566.5\mathbf{j} \quad \text{N}$$

It is obvious that the 400 N force in x-direction and the y-component of 250 N cause no moment about G. Therefore the moment at point G is

$$M_G = 350 \times 1.65 + 250\sin 30 \times 0.9 = 690 \quad \text{Nm} \quad \text{CCW}$$

Note that the x-component of the resultant force does not contribute the moment about G. Hence only $\mathbf{R}_y$ will be used in the moment calculation, from

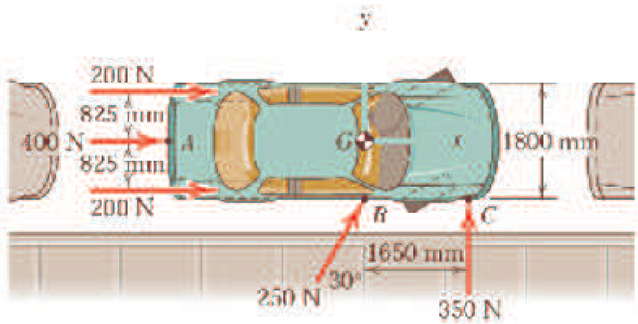


Figure 2.40: Example 2.19 ([1], pp. 65)

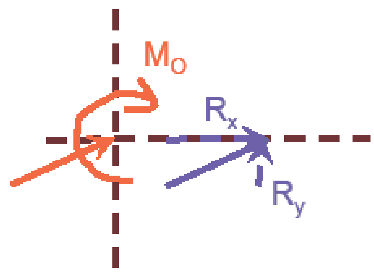


Figure 2.41: Solution to example 2.19

which the position on the centerline of the car where the resultant passes can be deduced.

$$566.5 \times x = 690, \quad x = 1.218 \quad \text{m}$$

Example 2.20 ([1], Prob. 2/98) An exhaust system for a pickup truck is shown in the figure. The weights W_h, W_m, and W_t of the headpipe, muffler, and tailpipe are 10, 100, and 50 N, respectively, and act at the indicated points. If the exhaust pipe hanger at point A is adjusted so that its tension F_A is 50 N, determine the required forces in the hangers at point B, C, and D so that the force-couple system at point O is zero. Why is a zero force-couple system at O desirable.

Solution: The key is that for static, if the force-couple at point O is zero, the force-couple at any point is zero too! In particular, the moment at point E is zero. That is

$$W_h \times (0.2 + 1.3 + 0.9) + W_m \times (0.65 + 0.9) + W_t \times 0.4 - F_A \times (1.3 + 0.9) - F_B \times 0.9 = 0$$

$$F_B = 98.9 \quad \text{N}$$

We use the knowledge that the force components in horizontal and vertical direction are zero to determine the remaining unknown forces, F_C and F_D.

$$F_A + F_B + F_C \cos 30 + F_D \cos 30 - W_h - W_m - W_t = 0$$

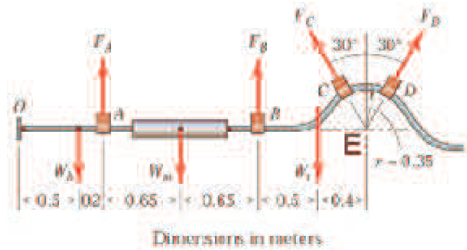


Figure 2.42: Example 2.20 ([1], pp. 65)

$$F_D \sin 30 - F_C \sin 30 = 0$$

$$F_C = F_D = 6.415 \quad \text{N}$$

Therefore the pipe is in equilibrium without external reaction force at support O. The stress at O, which is at the joint, is zero. This effectively prevents the breakage of the exhaust system.

2.5 3-D Rectangular Coordinate Systems

Figure 2.43 shows the force vector and its rectangular components. If a 3-D rectangular coordinate system has been specified, a force vector $\mathbf{F}$ can be written as the addition of its component vectors along the coordinate axes.

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{2.14}$$

The components are the orthogonal projection of the vector onto the respective axes which are determined by the dot product of the vector and the unit vector along the axes.

$$\begin{aligned} F_x &= \mathbf{F}\cdot\mathbf{i} = F\cos\theta_x \\ F_y &= \mathbf{F}\cdot\mathbf{j} = F\cos\theta_y \\ F_z &= \mathbf{F}\cdot\mathbf{k} = F\cos\theta_z \end{aligned} \tag{2.15}$$

The magnitude and direction of the force vector $\mathbf{F}$ follow immediately as

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{2.16}$$

$$\begin{aligned} \theta_x &= \arccos(F_x,\ F) \\ \theta_y &= \arccos(F_y,\ F) \\ \theta_z &= \arccos(F_z,\ F) \end{aligned} \tag{2.17}$$

And the following relationship can be seen directly from the figure;

$$\begin{aligned} F_{xz} &= F\sin\theta_y \\ F_{xy} &= F\sin\theta_z \\ F_{yz} &= F\sin\theta_x \end{aligned} \tag{2.18}$$

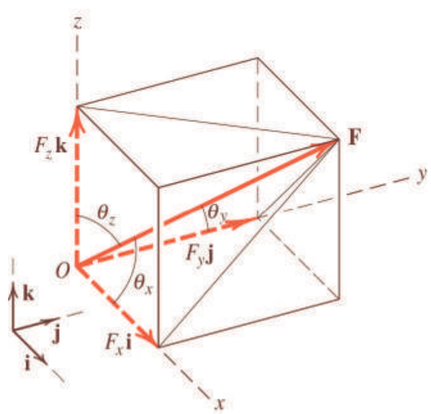


Figure 2.43: Force vector and its rectangular components ([1], pp. 66)

We define the directional unit vector, $\mathbf{n_F}$ to be the unit vector of which its direction is the same as the vector $\mathbf{F}$:

$$\mathbf{n_F} = \cos\theta_x \mathbf{i} + \cos\theta_y \mathbf{j} + \cos\theta_z \mathbf{k} \tag{2.19}$$

Hence $\mathbf{F}$ can also be represented as

$$\mathbf{F} = F\mathbf{n_F} \tag{2.20}$$

That is to completely describe the vector, it is necessary to determine the directional unit vector and the magnitude. In many problems, there are common situations we would like to determine the direction of the force vector: direction of force vector by *two points* or by *two angles*.

Direction of force vector by two points

If we were given two points, A and B, by which the force vector, $\mathbf{F}$ passes, the directional unit vector, $\mathbf{n_F}$, can be calculated as follow. See figure 2.44.

$$\mathbf{n_F} = \frac{AB}{|AB|} = \frac{(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} \tag{2.21}$$

Direction of force vector by two angles

If we were given two angles, θ and ϕ, which the force vector, $\mathbf{F}$ makes with the x-axis and the projection onto the x-y plane, respectively, $\mathbf{n_F}$ can be calculated as follow. See figure 2.45.

$$\mathbf{n_F} = (\cos\phi\cos\theta)\,\mathbf{i} + (\cos\phi\sin\theta)\,\mathbf{j} + (\sin\phi)\,\mathbf{k} \tag{2.22}$$

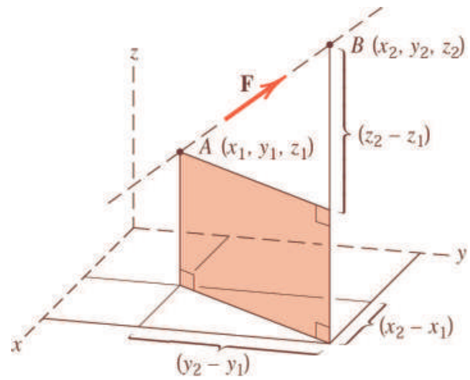


Figure 2.44: Direction of force vector by two points ([1], pp. 66)

Orthogonal projection

The magnitude of the orthogonal projection of **F** in the **n**-direction is determined by the dot product of **F** with **n**. Mathematically,

$$F_n = \mathbf{F} \cdot \mathbf{n} = F\mathbf{n_F} \cdot \mathbf{n} = F\cos\theta \tag{2.23}$$

where θ is the angle that **F** made with **n**. Consequently,

$$\cos\theta = \frac{\mathbf{F} \cdot \mathbf{n}}{F} \tag{2.24}$$

Therefore the orthogonal projection of **F** onto the **n**-direction is

$$\mathbf{F_n} = F_n\mathbf{n} = (\mathbf{F} \cdot \mathbf{n})\,\mathbf{n} \tag{2.25}$$

See figure 2.46.

It is noted that the orthogonal projection of the force vector *may not be equal* to its component. They will be equal only when the rectangular coordinate system is used.

Example 2.21 ([2], Prob. 2/99) In opening a door which is equipped with a heavy duty return mechanism, a person exerts a force **P** of magnitude 32 N as shown. Force **P** and the normal **n** to the face of the door lie in a vertical plane. Express **P** as a vector and determine the angles θ_x, θ_y, and θ_z which the line of action **P** makes with the positive x-, y-, and z-axes.

Solution: Force **P** can be described in x-y-z frame by first projecting **P**

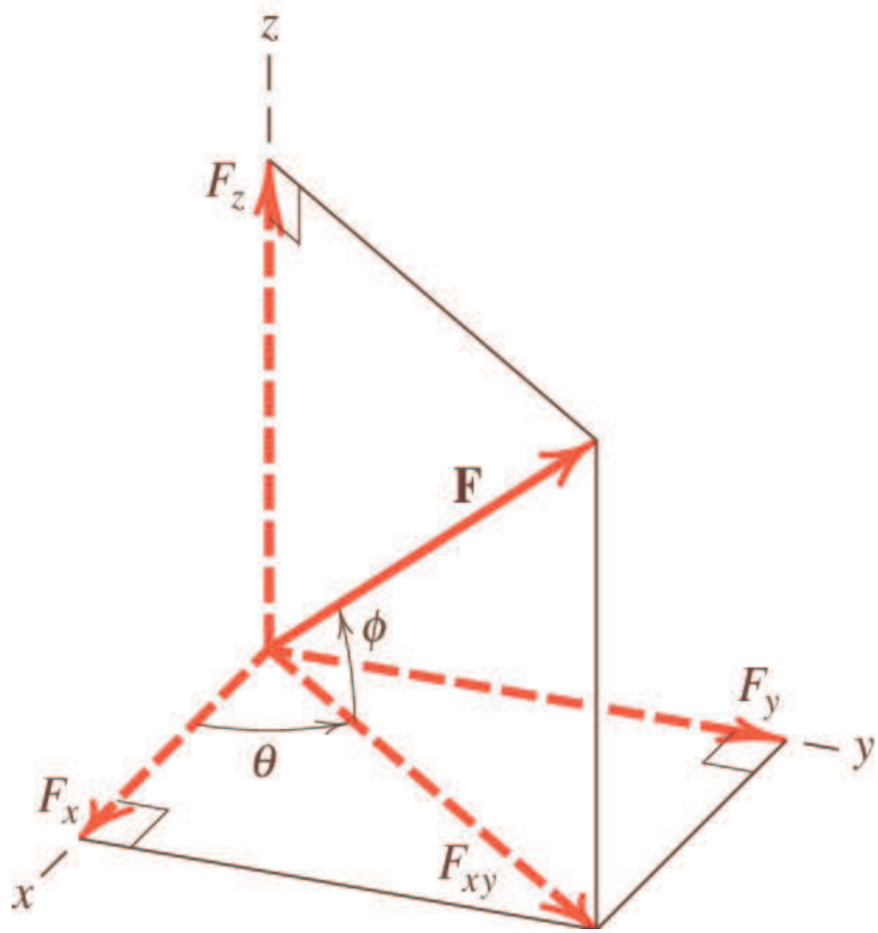


Figure 2.45: Direction of force vector by two angles ([1], pp. 67)

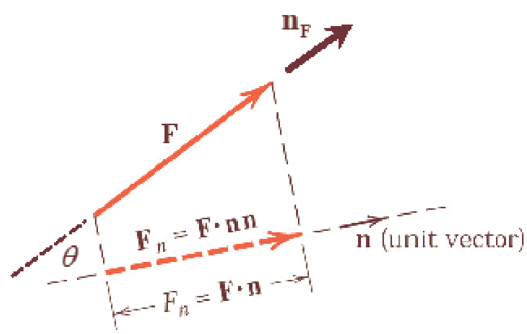


Figure 2.46: Orthogonal projection of **F** onto the **n**-direction ([1], pp. 67)

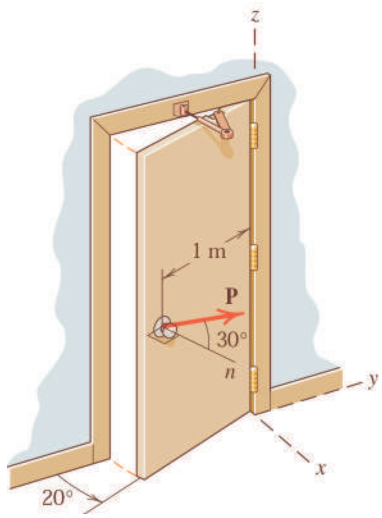


Figure 2.47: Example 2.21 ([2], pp. 68)

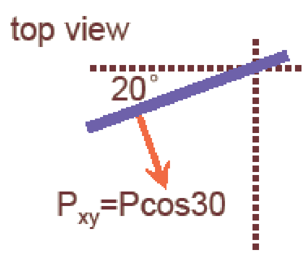


Figure 2.48: Solution to example 2.21

onto the z-axis and the x-y plane. Then the force in the x-y plane is further decomposed into the x- and y- axes, respectively.

$$\begin{aligned} \mathbf{P} &= P\cos 30\cos 20\mathbf{i} + P\cos 30\sin 20\mathbf{j} + P\sin 30\mathbf{k} \\ &= 26.0\mathbf{i} + 9.48\mathbf{j} + 16\mathbf{k} \quad \text{N} \end{aligned}$$

The directional angle is determined by eq. 2.24.

$$\begin{aligned} \theta_x &= \text{acos}\left(\frac{\mathbf{P}\cdot\mathbf{i}}{P}\right) = 34.5^\circ \\ \theta_y &= \text{acos}\left(\frac{\mathbf{P}\cdot\mathbf{j}}{P}\right) = 72.8^\circ \\ \theta_z &= \text{acos}\left(\frac{\mathbf{P}\cdot\mathbf{k}}{P}\right) = 60^\circ \end{aligned}$$

Example 2.22 ([1], Prob. 2/112) The rectangular plate is supported by hinges along its side BC and by the cable AE. If the cable tension is 300 N, determine the projection onto line BC of the force exerted on the plate by the cable.

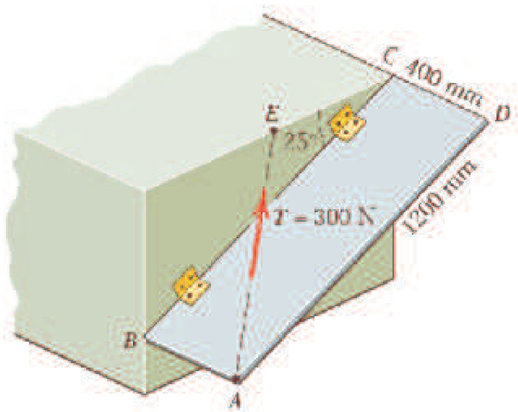


Figure 2.49: Example and solution of 2.22 ([1], pp. 72)

Note that E is the midpoint of the horizontal upper edge of the structural support.

Solution: This problem is best solved by first determining the pertinent directional unit vectors. Next we describe the tension $\mathbf{T}$ and the unit vector $\mathbf{n}_{BC}$ onto which it will be projected. Finally, the projection is found by calculating the dot product of the tension with that unit vector. For each point,

$$A = (-0.4, 0, 1.2\sin 25) \quad B = (0, 0, 1.2\sin 25) \quad C = (0, 1.2\cos 25, 0)$$

$$D = (-0.4, 1.2\cos 25, 0) \quad E = (0, 0.6\cos 25, 0)$$

The tension $\mathbf{T}$ and the unit vector $\mathbf{n}_{BC}$ is then readily determined:

$$\mathbf{T} = T\frac{AE}{|AE|} = 142.1\mathbf{i} + 193.2\mathbf{j} - 180.2\mathbf{k}$$

$$\mathbf{n}_{BC} = \frac{BC}{|BC|} = 0.9063\mathbf{i} - 0.4226\mathbf{k}$$

Therefore the projection of $\mathbf{T}$ onto $\mathbf{n}_{BC}$ is

$$T_{BC} = \mathbf{T}\cdot\mathbf{n}_{BC} = 251.2 \quad \text{N}$$

Example 2.23 ([2], Prob. 2/104) The power line is strung from the power-pole arm at A to point B on the same horizontal plane. Because of the sag of the cable in the vertical plane, the cable makes an angle of 15° with the horizontal where it attaches to A. If the cable tension at A is 800 N, write $\mathbf{T}$ as a vector and determine the magnitude of its projection onto the x-z plane.

Solution: Direction of the tension $\mathbf{T}$ is conveniently described by two angles. Imagine a line originally oriented along $\overline{CB}$. It is then rotated by the

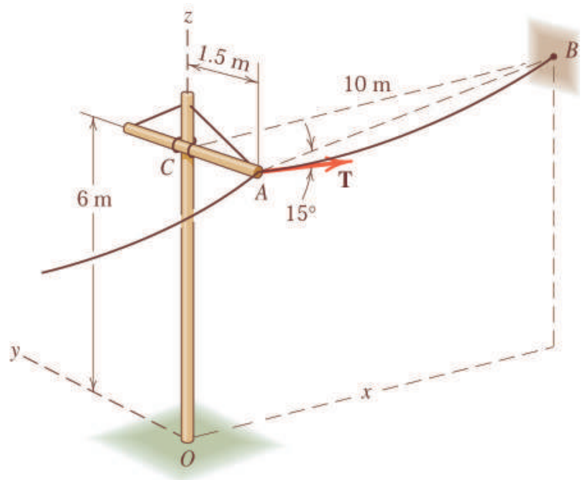


Figure 2.50: Example and solution of 2.23 ([2], pp. 69)

angle θ so the new orientation is along $\overline{AB}$. Finally the line is vertically rotated downward by 15° and hence its direction is the same as that of $\mathbf{T}$. With the known directional unit vector, we can write $\mathbf{T}$ in x-y-z frame. The magnitude of its projection onto the x-z plane is followed immediately.

The angle θ is

$$\theta = \text{atan}\,(1.5/10) = 8.53^\circ$$

Therefore the tension force $\mathbf{T}$ can be described as

$$\mathbf{T} = T\cos 15\cos\theta\mathbf{i} + T\cos 15\sin\theta\mathbf{j} - T\sin 15\mathbf{k} = 764.2\mathbf{i} + 114.6\mathbf{j} - 207\mathbf{k} \quad \text{N}$$

The magnitude of its projection onto the x-z is then

$$T_{xz} = \sqrt{T_x^2 + T_z^2} = 792 \quad \text{N}$$

It can also be determined by first calculating the angle that $\mathbf{T}$ made with the y-axis:

$$\theta_y = \text{acos}\left(\frac{\mathbf{T}\cdot\mathbf{j}}{T}\right) = 81.76^\circ$$

Referring to fig. 2.43 and eq. 2.18,

$$T_{xz} = T\sin\theta_y = 792 \quad \text{N}$$

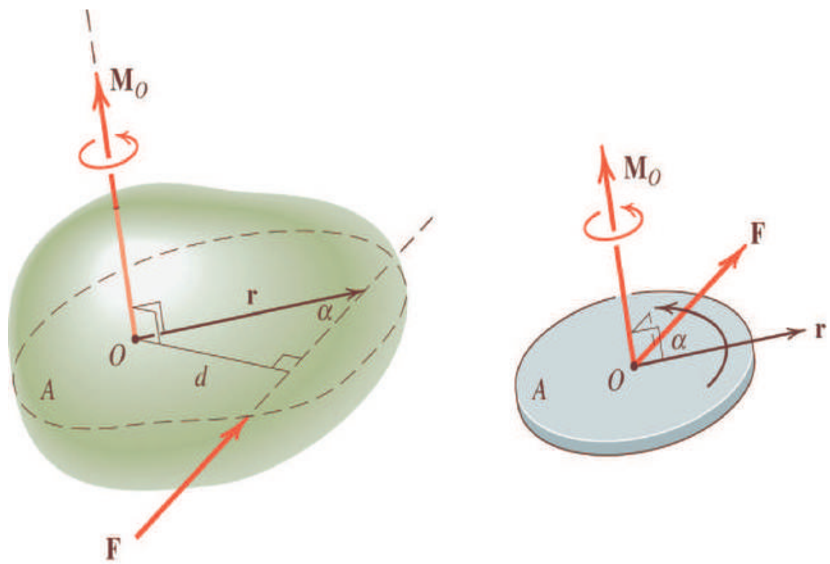


Figure 2.51: Moment $\mathbf{M}_O$ of $\mathbf{F}$ about O ([1], pp. 74)

2.6 3-D Force, Moment, and Couple

Calculation of the moment and couple in three dimension using the scalar approach is more awkward than using the vector approach. Figure 2.51 shows the relevant parameters. $\mathbf{r}$ is a vector originated from O to an arbitrary point on the line of the force vector $\mathbf{F}$. $\mathbf{M}_O$ is the moment of $\mathbf{F}$ about O. d is the length of the perpendicular line from O to the line of the force vector and α is the angle that $\mathbf{r}$ made with $\mathbf{F}$. Consequently, the moment of $\mathbf{F}$ about O is

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{2.26}$$

From the equation, it can be concluded that $\mathbf{M}_O \perp \mathbf{r}$ and $\mathbf{M}_O \perp \mathbf{F}$. Also, $\mathbf{M}_O$ is through O and is normal to the plane made by $\mathbf{r}$ and $\mathbf{F}$.

$\mathbf{M}_O$ may be determined by writing $\mathbf{r}$ and $\mathbf{F}$ in the rectangular coordinate frame and performing the vector cross product;

$$\begin{aligned} \mathbf{M}_O &= \mathbf{r} \times \mathbf{F} = (r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}) \times (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\ &= (r_yF_z - r_zF_y)\,\mathbf{i} + (r_zF_x - r_xF_z)\,\mathbf{j} + (r_xF_y - r_yF_x)\,\mathbf{k} \end{aligned} \tag{2.27}$$

This formula can be remembered easily by recognizing the expression as the following determinant;

$$\mathbf{r} \times \mathbf{F} \triangleright \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{2.28}$$

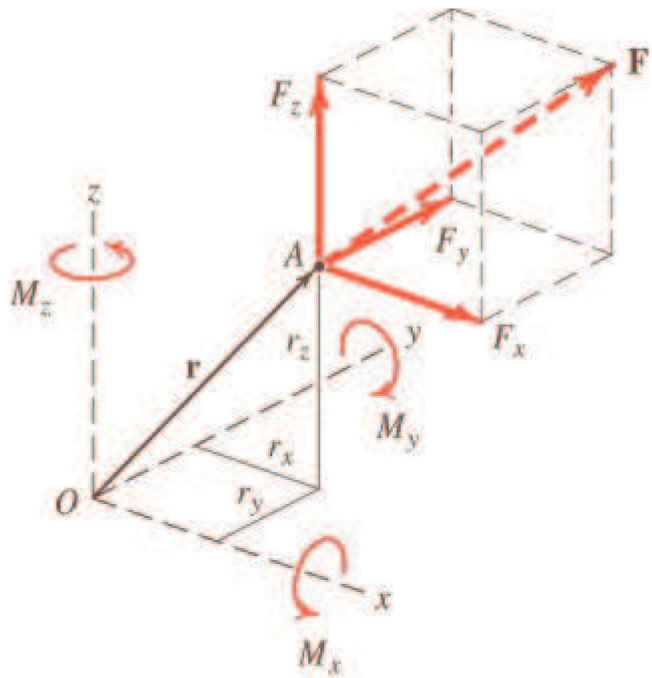


Figure 2.52: Moment determination by the principle of moment ([1], pp. 75)

We can proofcheck this vector cross product and gain understanding in the process of determining the moment by applying the principle of moment. According to fig 2.52,

$$\begin{aligned} M_x &= r_y F_z - r_z F_y \\ M_y &= r_z F_x - r_x F_z \\ M_z &= r_x F_y - r_y F_x \end{aligned} \tag{2.29}$$

which confirms the mathematical operation.

2.6.1 Moment about an axis

Sometimes it is necessary to know the moment $\mathbf{M}_\lambda$ of $\mathbf{F}$ about an axis λ through O. See fig. 2.53. Essentially $\mathbf{M}_\lambda$ is the component of $\mathbf{M}_O$ along the λ-axis. Therefore the following steps are used to determine $\mathbf{M}_\lambda$.

1. Determine moment $\mathbf{M}_O$ of $\mathbf{F}$ about O: $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.

2. Orthogonally project $\mathbf{M}_O$ onto the $\mathbf{n}$-direction along the axis λ. That is

$$\mathbf{M}_\lambda = (\mathbf{M}_O \cdot \mathbf{n})\,\mathbf{n} = (\mathbf{r} \times \mathbf{F} \cdot \mathbf{n})\,\mathbf{n} \tag{2.30}$$

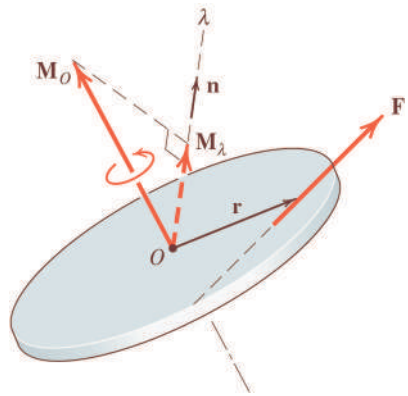


Figure 2.53: Moment of $\mathbf{F}$ about the axis λ ([1], pp. 75)

From mathematics, the magnitude of $\mathbf{M}_\lambda$ is

$$M_\lambda = \begin{vmatrix} r_x & r_y & r_z \\ F_x & F_y & F_z \\ n_x & n_y & n_z \end{vmatrix} \tag{2.31}$$

There is an interesting point of the moment about an axis. Point O can be *any* point on the axis λ and it still gives equal value of $\mathbf{M}_\lambda$ even, of course, $\mathbf{M}_O$ is distinct for each of the chosen point O.

2.6.2 3-D couple

Analogous to 2-D situation, two forces with equal magnitude but pointing in the opposite direction and with non-zero offset will induce the couple. Its direction is perpendicular to the plane made with the line of action of the forces. See figure 2.54.

Evaluation of the couple is straightforward. First determine the moment of each of the force about an arbitrary point O. Sum of these two moments will be the couple of the pair of forces. This is also equivalent to the cross product of an arbitrary position vector $\mathbf{r}$, connecting the pair of forces, with the force vector it is pointing to.

Therefore the couple is a free vector and has no associated point. Moreover the couple can be helpful in adding pair of forces. Pair of forces $\{\mathbf{F_1}, -\mathbf{F_1}\}$ and $\{\mathbf{F_2}, -\mathbf{F_2}\}$ can be added up by noting they are equivalent to the couple $\mathbf{M}_1$ and $\mathbf{M}_2$. Since couples are free vectors, they can move and add up using the parallelogram law so long as the direction is maintained. Result is the couple $\mathbf{M}$, which is then decomposed back into a pair of forces $\{\mathbf{F}, -\mathbf{F}\}$.

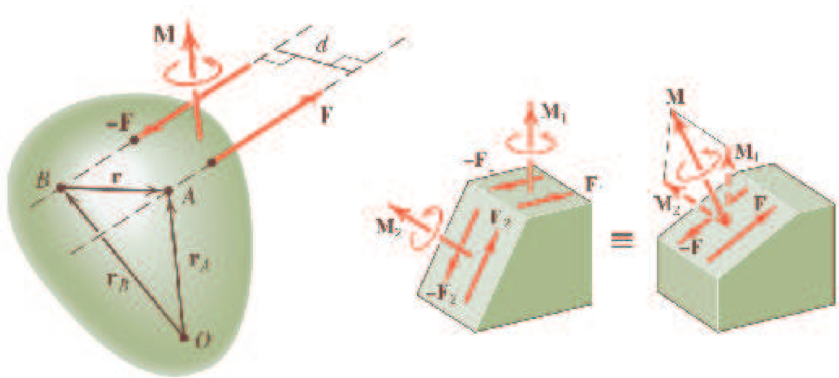


Figure 2.54: Three dimensional couple ([1], pp. 76)

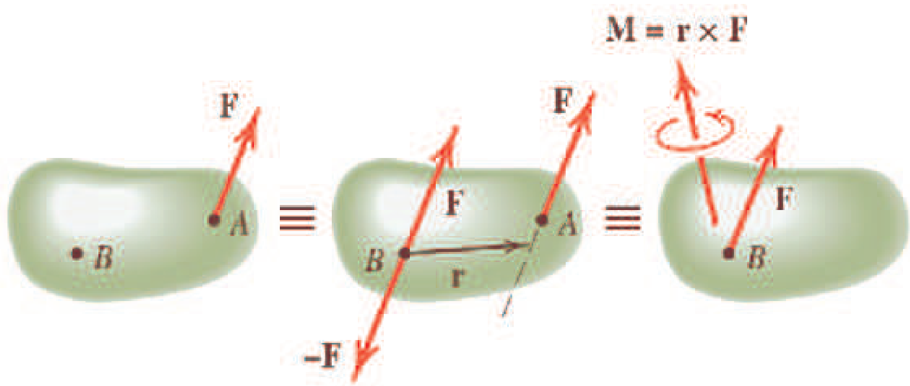


Figure 2.55: Three dimensional equivalent force-couple system ([1], pp. 77)

2.6.3 3-D equivalent force-couple system

The procedure in 2-D problem can be readily extended to cope with 3-D problem. To maintain the external effect of moving the force to a new location, the compensated couple must be provided. Specifically,

1. Specify the destination point B.
2. Move the force so the new line of action pass through B. The new line are parallel to the original one. By the force-couple equivalent method, the couple must be generated to preserve the motion effect of the original force system. If $\mathbf{r}$ is the position vector from B to any point on the line of action of $\mathbf{F}$, the compensated couple is $\mathbf{M} = \mathbf{r} \times \mathbf{F}$.

Figure 2.55 illustrates the procedure.

Example 2.24 ([1], Prob. 2/123) The helicopter is drawn here with certain 3-D geometry given. During a ground test, a 400 N aerodynamic force is applied

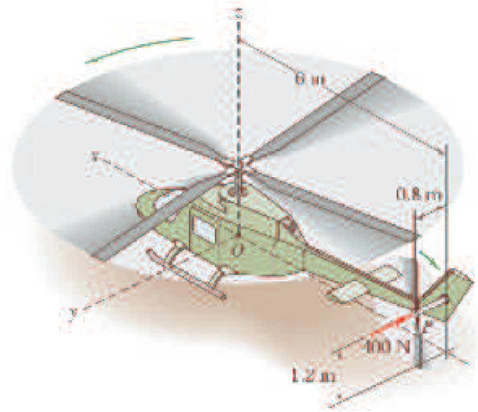


Figure 2.56: Example and solution of 2.24 ([1], pp. 82)

to the tail rotor at P as shown. Determine the moment of this force about point O of the airframe.

Solution: For this simple force $\mathbf{P}$, we can determine the moment component-wise. It is obvious that the force is in y-direction. Hence it does not cause moment in the y-direction. By observation,

$$\mathbf{M}_O = (400 \times 1.2)\,\mathbf{i} + (400 \times 6)\,\mathbf{k} = 480\mathbf{i} + 2400\mathbf{k} \quad \text{N}$$

Example 2.25 ([2], Prob. 2/118) In picking up a load from position A, a cable tension $\mathbf{T}$ of magnitude 21 kN is developed. Calculate the moment that $\mathbf{T}$ produces about the base O of the construction crane.

Solution: Vectorial approach is appropriate for this problem because the location of each point is explicitly given. According to the given coordinate frame,

$$A = (0, 18, 30) \quad B = (6, 13, 0)$$

The tension vector $\mathbf{T}$ is

$$\mathbf{T} = T\frac{\overline{AB}}{|AB|} = 4.06\mathbf{i} - 3.39\mathbf{j} - 20.32\mathbf{k} \quad \text{kN}$$

And the position vector $\mathbf{r}$ is

$$\mathbf{r} = \overline{OA} = 18\mathbf{j} + 30\mathbf{k} \quad \text{m}$$

Therefore, the moment of $\mathbf{T}$ about O is

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{T} = -264.2\mathbf{i} + 121.9\mathbf{j} - 73.2\mathbf{k} \quad \text{kNm}$$

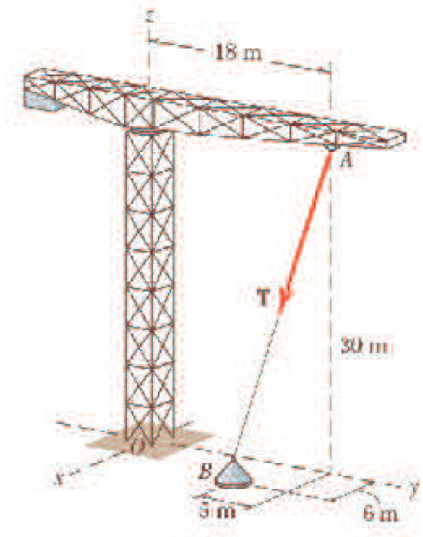


Figure 2.57: Example and solution of 2.25 ([2], pp. 80)

Example 2.26 ([2], Prob. 2/122) The special-purpose milling cutter is subjected to the force of 1200 N and a couple of 240 Nm as shown. Determine the moment of this system about point O.

Solution: First describe all quantities in the given coordinate system.

$$\mathbf{R} = 1200\cos 30\mathbf{j} - 1200\sin 30\mathbf{k} = 1039\mathbf{j} - 600\mathbf{k} \quad \text{N}$$

$$\mathbf{r} = 0.2\mathbf{i} + 0.25\mathbf{k} \quad \text{m}$$

The moment of this system about O is the summation of the moment induced by the force and the free vector couple;

$$\begin{aligned} \mathbf{M}_O &= \mathbf{r} \times \mathbf{R} + 240\cos 30\mathbf{j} - 240\sin 30\mathbf{k} \\ &= -259.8\mathbf{i} + 327.8\mathbf{j} + 87.8\mathbf{k} \quad \text{Nm} \end{aligned}$$

Example 2.27 ([1], Prob. 2/141) A 5 N vertical force is applied to the knob of the window-opener mechanism when the crank BC is horizontal. Determine the moment of the force about point A and about the line AB.

Solution: As usual, describe the related quantities using the same coordinate frame.

$$\mathbf{r} = 75\cos 30\mathbf{i} + 75\mathbf{j} + 75\sin 30\mathbf{k} \quad \text{mm}$$

$$\mathbf{F} = -5\mathbf{k} \quad \text{N}$$

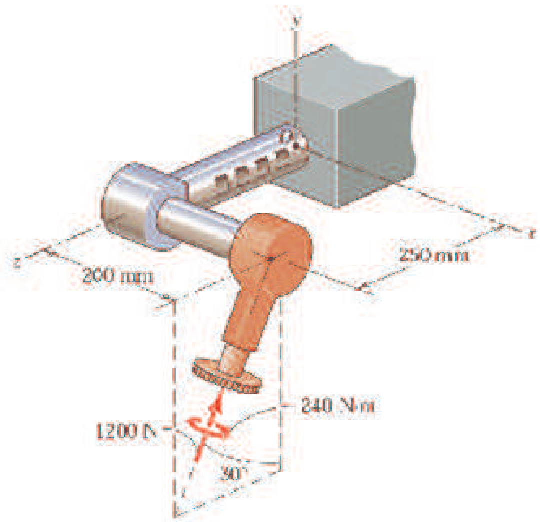


Figure 2.58: Example and solution of 2.26 ([2], pp. 81)

The moment of the force about point A is

$$\mathbf{M}_A = \mathbf{r} \times \mathbf{F} = -375\mathbf{i} + 325\mathbf{j} \quad \text{Nmm}$$

The moment about the line AB is then the component of $\mathbf{M}_A$ about AB, which can be found by the dot product.

$$\mathbf{n}_{AB} = \cos 30\mathbf{i} + \sin 30\mathbf{k}$$

$$\mathbf{M}_{AB} = (\mathbf{M}_A \cdot \mathbf{n}_{AB})\,\mathbf{n}_{AB} = -281\mathbf{i} - 162.4\mathbf{k} \quad \text{Nmm}$$

2.7 3-D Resultants

Resultant is the simplest force combination which can replace the original system of forces, moments, and couples without altering the external effect of the system on the rigid body. As mentioned earlier, the vectorial approach is more suitable to the 3-D problems. Here the procedure in determining the resultant is explained.

1. Define the suitable rectangular coordinate system and specify a convenient point O about which the moment will be determined.

2. Move all forces so the new lines of action pass through point O. This step is already explained in the force-couple equivalence topic.

3. Sum all the forces to $\mathbf{R}$ and sum all the couples to $\mathbf{M}$.

$$\mathbf{R} = \sum \mathbf{F} \tag{2.32}$$

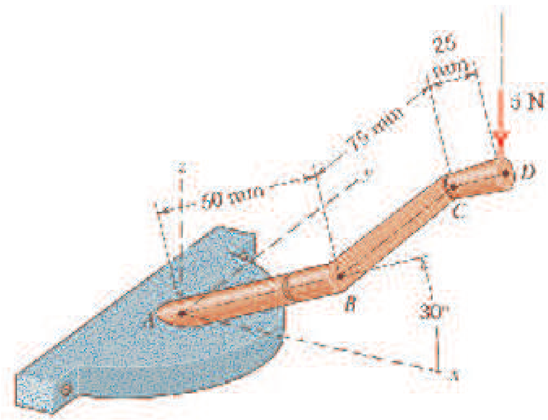


Figure 2.59: Example and solution of 2.27 ([1], pp. 87)

$$\mathbf{M} = \sum (\mathbf{r} \times \mathbf{F}) \tag{2.33}$$

4. Locate the correct line of action of the resultant **R**. This step is the reverse of the force-couple equivalence. The goal is to solve for the piercing point of the resultant with the object. In 3-D problem, it involves determining 2 unknowns in the moment equation, $\mathbf{r} \times \mathbf{R} = \mathbf{M}$. The matrix equation is of rank 2. That is one of the equation is degenerated.

Figure 2.60 captures the process.

The selected point O, which can be any convenient point, specifies the value of the couple **M**. This is because in statics, we can set up the equilibrium equations

$$\sum \mathbf{F} = \mathbf{O} \tag{2.34}$$

$$\sum \mathbf{M} = \mathbf{0} \tag{2.35}$$

at virtually any point. However, in dynamics the equations of motion will be greatly simplified at some special points, such as the fixed point or the point of center of gravity (CG). The equations at CG are

$$\sum \mathbf{F} = m\ddot{\mathbf{x}}_G \tag{2.36}$$

$$\sum \mathbf{M}_G = I_G\ddot{\theta} \tag{2.37}$$

Therefore in dynamics it is usually necessary to calculate the resultants at CG, which is the selected point O in our context.

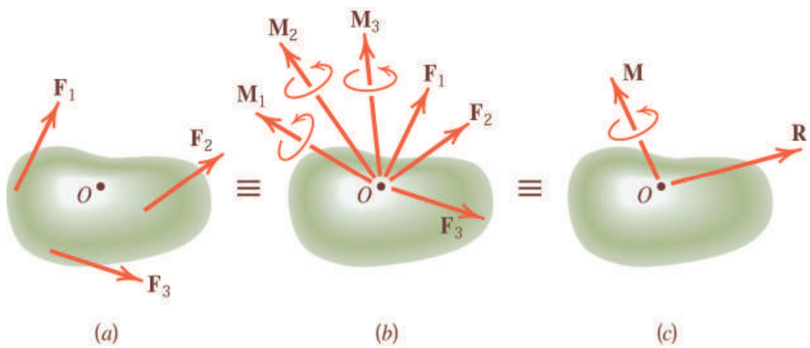


Figure 2.60: Determination of the resultant force ([1], pp. 88)

2.7.1 Resultants of special force systems

Concurrent forces If all forces are concurrent, there will be no moment about the point of concurrency. That is,

$$\mathbf{R} = \sum \mathbf{F} \tag{2.38}$$

and

$$\mathbf{M} = \sum (\mathbf{r} \times \mathbf{F}) = \mathbf{0} \tag{2.39}$$

Parallel forces If the lines of action of all forces are parallel, the resultant force can be obtained algebraically. The magnitude of the resultant force, $\mathbf{R}$, is the magnitude of the algebraic sum of the given forces.

$$|\mathbf{R}| = \left|\sum F\right| \tag{2.40}$$

And the moment will be perpendicular to the parallel lines of action.

$$\mathbf{M}_O = \sum \mathbf{M} = \mathbf{r} \times \mathbf{R} \tag{2.41}$$

Wrench resultant If the resultant force and the resultant moment, as shown in fig. 2.61, the resultant system is called *wrench resultant*. An example is the reaction force and moment of the screwdriver. If the force and moment are in the same direction, it is called *positive wrench*. Otherwise it is the *negative wrench*.

Wrench resultant is the simplest form to visualize the effect of general force system on to the object. That is the object is simultaneously translating and rotating about the unique axis: the screw axis.

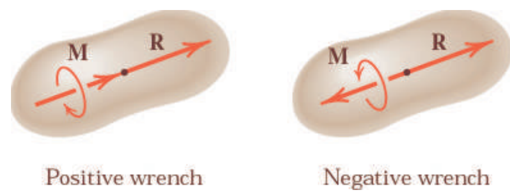


Figure 2.61: A special force system: wrench system ([1], pp. 89)

2.7.2 Wrench resultant – Force-Couple equivalence

Any force system has the equivalent wrench resultant. Following is the procedure in determining the wrench resultant. Figure 2.62 graphically shows each step.

1. Determine the force-couple resultant $\mathbf{R}$ and $\mathbf{M}$ at any convenient point O.
2. Orthogonally project $\mathbf{M}$ along and perpendicular to the line of action of $\mathbf{R}$, $\mathbf{n_R}$. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the components of $\mathbf{M}$ along and perpendicular to $\mathbf{n_R}$, respectively. Then,
$$\mathbf{n_R} = \frac{\mathbf{R}}{|\mathbf{R}|} \tag{2.42}$$
$$\mathbf{M}_1 = (\mathbf{M} \cdot \mathbf{n_R})\,\mathbf{n_R} \tag{2.43}$$
$$\mathbf{M}_2 = \mathbf{M} - \mathbf{M}_1 \tag{2.44}$$
3. Transform the couple $\mathbf{M}_2$ into the equivalent pair of $\mathbf{R}$ and $-\mathbf{R}$, with $-\mathbf{R}$ applied at O to cancel $\mathbf{R}$, as shown in fig. 2.62c.
4. Since the couple is a free vector, we can move $\mathbf{M}_1$, which is parallel to $\mathbf{R}$, such that its line of action is the same as that of $\mathbf{R}$. The result is the wrench resultant with the correct line of action.

Example 2.28 ([1], Prob. 2/148) The pulley and gear are subjected to the loads shown. For these forces, determine the equivalent force-couple system at point O.

Solution: This is the typical step in shaft analysis. First we determine the resultant force $\mathbf{R}$. The resultant moment $\mathbf{M}_O$ is determined by the help of principle of moment and by inspection, since the force system is rather simple.

$$\mathbf{R} = (800 + 200 - 1200\sin 10)\,\mathbf{i} + 1200\cos 10\mathbf{j} = 792\mathbf{i} + 1182\mathbf{j} \quad \text{N}$$

Moment at O by 800 N force:

$$\mathbf{M}_1 = -800 \times 0.55\mathbf{j} - 800 \times 0.1\mathbf{k}$$

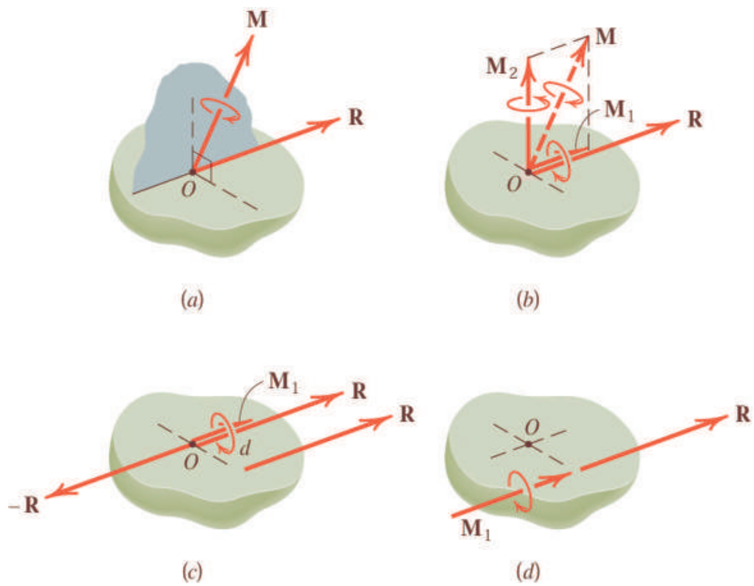


Figure 2.62: Wrench resultant determination ([1], pp. 90)

Moment at O by 200 N force:

$$\mathbf{M}_2 = -200 \times 0.55\mathbf{j} + 200 \times 0.1\mathbf{k}$$

Moment at O by 1200 N force:

$$\mathbf{M}_3 = 1200 \sin 10 \times 0.22\mathbf{j} + 1200 \cos 10 \times 0.075\mathbf{k} + 1200 \cos 10 \times 0.22\mathbf{i}$$

Therefore the resultant moment is the sum of the moments of three forces:

$$\mathbf{M}_O = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = 260\mathbf{i} - 504\mathbf{j} + 28.6\mathbf{k} \quad \text{Nm}$$

Example 2.29 ([1], Prob. 2/152) Two upward loads are exerted on the small 3D truss. Reduce these two loads to a single force-couple system at point O. Show that $\mathbf{R}$ is perpendicular to $\mathbf{M}_O$. Then determine the point in the x-z plane through which the resultant passes.

Solution: Since the moment involves the cross product of the position and force vector, $\mathbf{R}$ must be perpendicular to $\mathbf{M}_O$. However we have to show it explicitly for this problem.

$$\mathbf{R} = 800\mathbf{j} + 1600\mathbf{j} = 2400\mathbf{j} \quad \text{N}$$

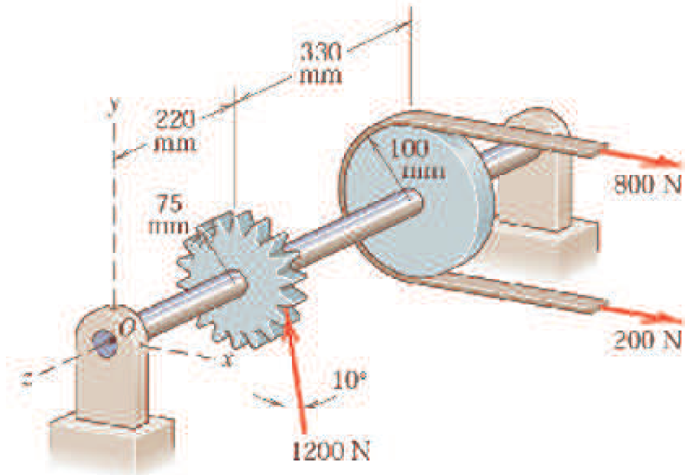


Figure 2.63: Example and solution of 2.28 ([1], pp. 95)

By inspection,

$$\mathbf{M}_O = 800 \times 2.4\mathbf{k} + 1600 \times 2.4\mathbf{k} + 1600 \times 0.9\mathbf{i} = 1440\mathbf{i} + 5760\mathbf{k} \quad \text{Nm}$$

It is therefore obvious that $\mathbf{R}$ is perpendicular to $\mathbf{M}_O$.

To determine the point in the x-z plane through which the resultant passes, we can use the observation since $\mathbf{R}$ is a simple force aligning with one of the coordinate axis. Note $\mathbf{R}$ is parallel to the y-axis and must produce the moment of 5760$\mathbf{k}$ Nm. Therefore $\mathbf{R}$ must be x m far from the y-z plane to produce 5760$\mathbf{k}$ Nm. That is

$$5760 = 2400 \times x, \quad x = 2.4 \quad \text{m}$$

And $\mathbf{R}$ must also produce the moment of 1440$\mathbf{i}$ Nm. Therefore $\mathbf{R}$ must be $-z$ m far from the x-y plane to produce 1440$\mathbf{i}$ Nm. That is

$$1440 = 2400 \times (-z)\,, \quad z = -0.6 \quad \text{m}$$

Consequently, the resultant force $\mathbf{R}$ passes through (2.4, −0.6) m in the x-z plane.

Example 2.30 ([2], Prob. 2/141) Replace the two forces acting on the block by a wrench. Write the moment $\mathbf{M}$ associated with the wrench as a vector and specify the coordinates of the point P in the x-y plane through which the line of action of the wrench passes.

Solution: In this problem, we will follow the procedure of determining the wrench resultant explained in section 2.7.2. First, we choose point O at which to determine the force-couple resultant. Hence,

$$\mathbf{R} = F\mathbf{i} - F\mathbf{k}$$

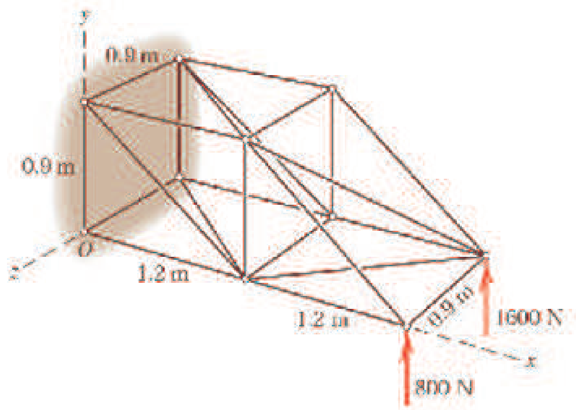


Figure 2.64: Example and solution of 2.29 ([1], pp. 96)

$$\mathbf{M}_O = F\left(a+c\right)\mathbf{j} - Fb\mathbf{k}$$

Next, we project $\mathbf{M}_O$ onto the direction parallel and perpendicular to $\mathbf{n_R}$, for which its value is

$$\mathbf{n_R} = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}$$

Consequently, the components of $\mathbf{M}_O$ are

$$\mathbf{M}_{\|} = \left(\mathbf{M}_O \cdot \mathbf{n_R}\right)\mathbf{n_R} = \frac{Fb}{2}\mathbf{i} - \frac{Fb}{2}\mathbf{k}$$

$$\mathbf{M}_{\perp} = \mathbf{M}_O - \mathbf{M}_{\|} = -\frac{Fb}{2}\mathbf{i} + F\left(a+c\right)\mathbf{j} - \frac{Fb}{2}\mathbf{k}$$

Thirdly, we transform the couple $\mathbf{M}_{\perp}$ into pair of forces $\mathbf{R}$ and $-\mathbf{R}$. Suppose $\mathbf{r}$ is the position vector of the desired piercing point in the x-y plane which can be described as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

The moment equation is formed to transform between the pair of forces and the couple;

$$\mathbf{M}_{\perp} = \mathbf{r} \times \mathbf{R}$$

in which the values of $\mathbf{M}_{\perp}$ and $\mathbf{R}$ are substituted to solve for $\mathbf{r}$. The coordinates are

$$x = a + c, \quad y = b/2$$

Finally, the wrench consists of $\mathbf{R}$ and $\mathbf{M}_{\|}$ which pass through the x-y plane at $x = a + c$ and $y = b/2$.

Example 2.31 ([1], Prob. 2/159) The resultant of the two forces and couple may be represented by a wrench. Determine the vector expression for the

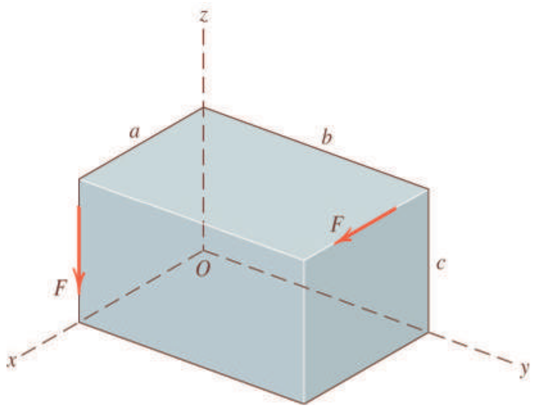


Figure 2.65: Example and solution of 2.30 ([2], pp. 94)

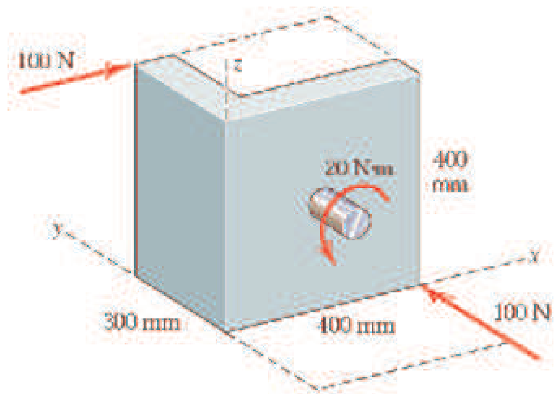


Figure 2.66: Example and solution of 2.31 ([1], pp. 98)

moment **M** of the wrench and find the coordinates of the point P in the x-z plane through which the resultant force of the wrench passes.

Solution: The resultant force is just simply

$$\mathbf{R} = 100\mathbf{i} + 100\mathbf{j} \;\; \text{N}$$

Another way to determine the wrench is to assume the point where the wrench passes. Let point P in the x-z plane, where the wrench passes, has the coordinate $(x, 0, z)$. Consequently, the moment of the force system about P is

$$\begin{aligned} \mathbf{M}_P &= 100 \times z\mathbf{i} + 100 \times (0.4 - x)\,\mathbf{k} + 100 \times (0.4 - z)\,\mathbf{j} - 100 \times 0.3\mathbf{k} - 20\mathbf{j} \\ &= 100z\mathbf{i} + (20 - 100z)\,\mathbf{j} + (10 - 100x)\,\mathbf{k} \;\; \text{Nm} \end{aligned}$$

Note that this moment at P must be equal to the couple of the wrench passing through P, which is parallel to the resultant force. That is $\mathbf{M}_P \parallel \mathbf{R}$. By comparing the ratio of their components, we have

$$\frac{100}{100z} = \frac{100}{20 - 100z}$$

and

$$10 - 100x = 0$$

As the result,

$$x = 0.1 \text{ m}, z = 0.1 \text{ m}$$

and substituting back into the moment equation, we have

$$\mathbf{M}_P = 10\mathbf{i} + 10\mathbf{j} \text{ Nm}$$

Chapter 3

Equilibrium

3.1 Introduction

Newton's first law states about the condition of a *particle* to be at rest or to be moving with a constant velocity. This is called the equilibrium condition. Later, physicists have extended the equilibrium condition to a *body*.

When a body is in equilibrium, the resultant on the body is zero. And if the resultant on a body is zero, the body is in equilibrium. Hence the necessary and sufficient conditions for equilibrium of a body are

$$\sum \mathbf{F} = \mathbf{O} \tag{3.1}$$

and

$$\sum \mathbf{M} = \mathbf{0} \tag{3.2}$$

These equations reveal that it is the prerequisite to determine the resultant force and moment acting on the body before the equilibrium condition can be applied. The method of free body diagram, explained in the next section, is a tool which helps in determining the correct resultant.

3.2 Mechanical System Isolation (FBD)

Free body diagram (FBD) is the most important first step in the mechanics problems. It defines clearly the interested system to be analyzed. It reveals *all* forces which act *on* the system. The system may be rigid, nonrigid, or their combinations. The system may be in fluid, gaseous, solid, or their combinations.

FBD represents the isolated or combination of bodies as a single body. After the FBD has been drawn, the corresponding indicated forces may be

1. *contact force* with other bodies that are removed virtually
2. *body force* such as gravitational or magnetic attraction forces

Figure 3.1 and 3.2 show common examples of modeling the action of forces in two dimensional problems. Beware that these are not the FBDs. Only the specific action forces, after the surroundings have been removed, are shown. Here are some explanation of each particular case.

1. Force by a flexible cable is always the tension force. Weight of the cable may be significant and hence make the cable sags.
2. Ideally, smooth surface cannot support the tangential or frictional force. For the rough surface, contact force may not necessary be normal to the tangential surface.

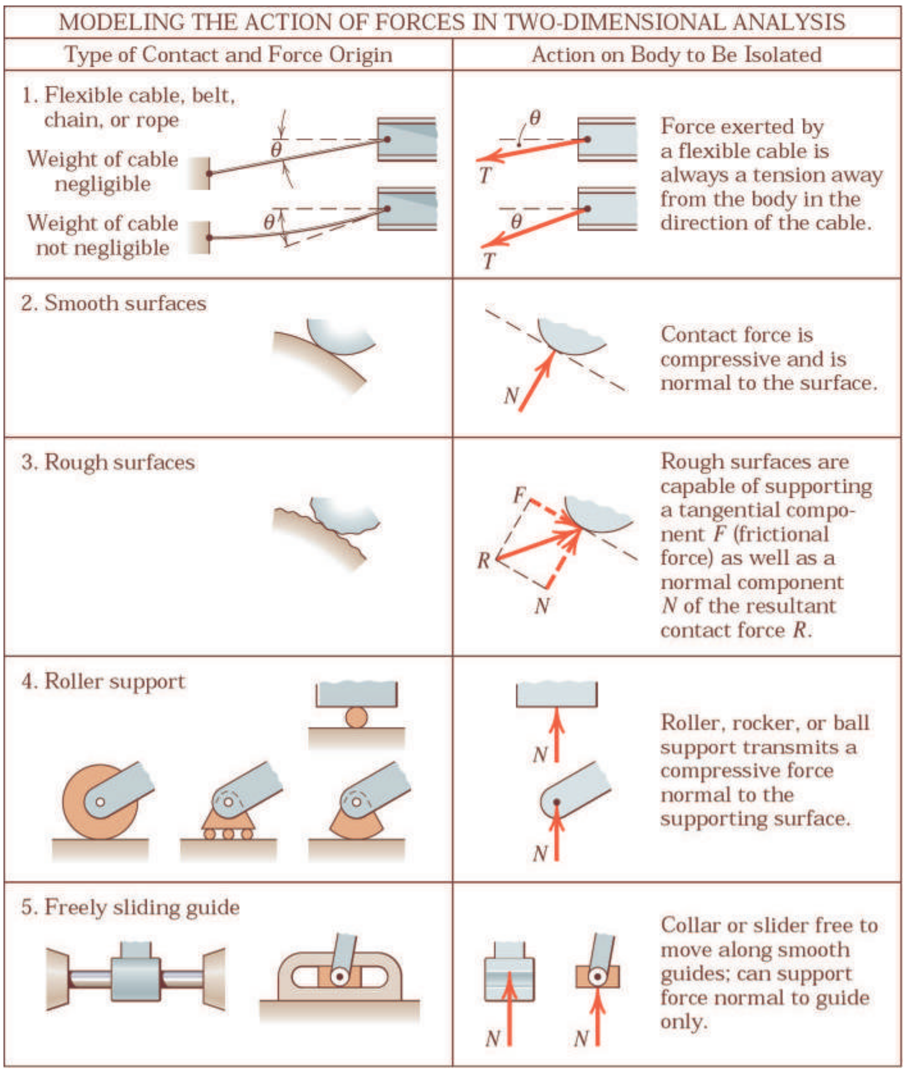
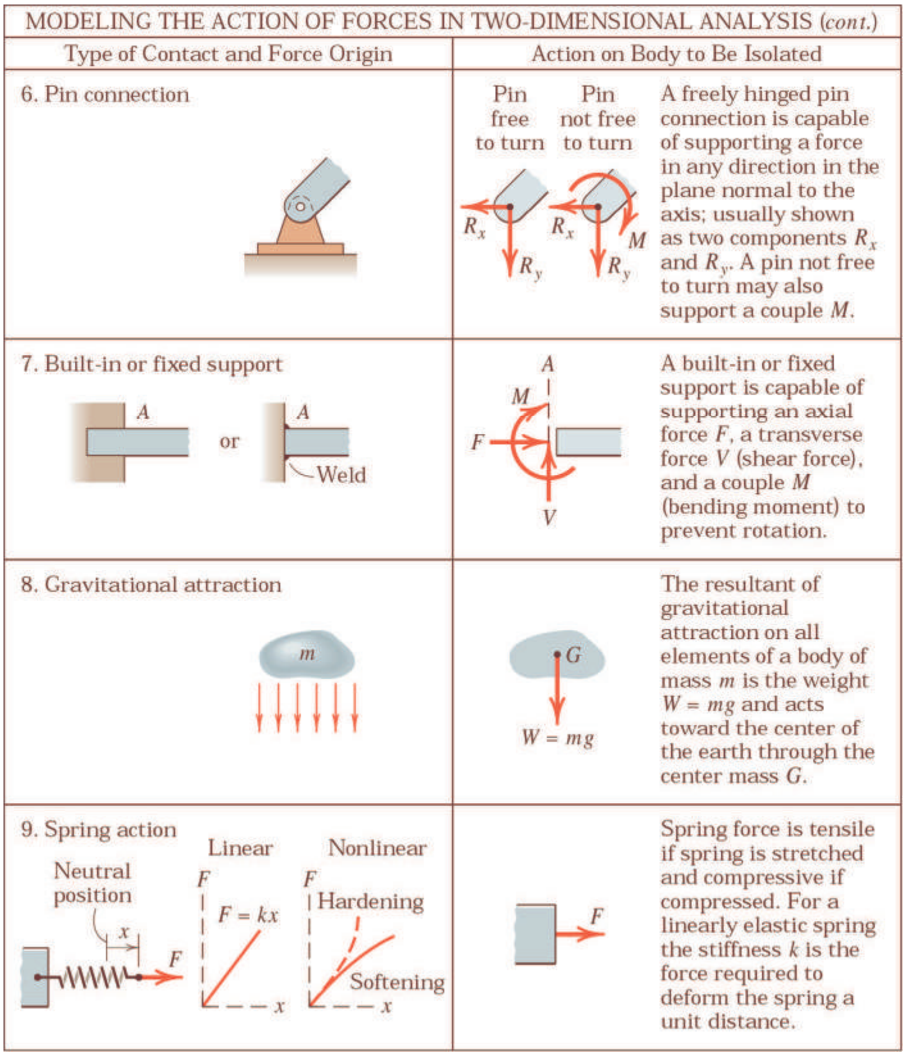


Figure 3.1: Common action of forces in two dimensional analysis ([1], pp. 111)



MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
6. Pin connection	Pin free to turn; Pin not free to turn; R_x, R_y, M. A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the axis; usually shown as two components R_x and R_y. A pin not free to turn may also support a couple M.
7. Built-in or fixed support (A or A, Weld)	A, M, F, V. A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction (m)	G, $W = mg$. The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action (Neutral position, x, F; Linear: F, $F = kx$, x; Nonlinear: F, Hardening, Softening, x)	F. Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Figure 3.2: Common action of forces in two dimensional analysis (continued) ([1], pp. 112)

3. Ideally, roller, rocker, smooth guide, or slider eliminate the frictional force. That is the supports cannot provide the resistance to motion in the tangential direction.

4. Pin connection provides support force in any direction normal to the pin axis. If the joint is not free to turn, a resisting couple may also be supported.

5. The built-in or fixed support of the beam is capable of supporting the axial force, the shear force, and the bending moment.

6. Gravitational force is a kind of distributed non-contact force. The resultant single force is the weight acted through the center of mass (C.M.) towards the center of the earth.

7. Remote action force has the same overall effects on a rigid body as direct contact force of equal magnitude and direction.

8. On the FBD, the force exerted *on* the body to be isolated *by* the body to be removed is indicated.

9. The sense of the force exerted *on* the FBD *by* the removed bodies opposes the movement which would occur if those bodies were removed.

10. If the correct sense cannot be known at first place, the sense of the scalar component is arbitrarily assigned. Upon computation, a negative algebraic sign indicates that the correct sense is opposite to that assigned.

After we have mastered indicating the action of forces, construction of the FBD is at eased. Here is a guideline in drawing the complete FBD.

1. Make decision which body or system is to be isolated. That system will usually involve the unknown quantities.

2. Draw *complete external boundary* of the system to *completely isolate* it from *all* other contacting or attracting bodies.

3. *All forces* that act *on* the isolated body *by* the removed contacting and attracting bodies are represented on the isolated body diagram. Forces should be indicated by vector arrows, each with its magnitude, direction, and sense. *Consistency* of the unknowns must be carried throughout the calculation.

4. Assign the convenient coordinate axes.

   "Only after the FBD is completed should the governing equations be applied"

There are some tips in writing the FBD.

1. Include as much as possible the system in FBD while the unknowns are still being revealed.

2. Internal forces to a rigid assembly of members do not influence the values of the external reactions. And so the external response of the mechanism as a whole would be unchanged.

3. Include the weights of the members on FBD.

4. Try to get the correct sense of the unknown vectors by visualizing the motion of the whole system when the supports are pretended to disappear. The correct sense will oppose the motion's direction.

5. Follow the action of force prototypes in determining the forces acted by the removed bodies.

Figure 3.3 shows some typical examples of drawing the FBD. In the first example, the internal forces of the structure are not seen because the boundary of the FBD encloses the entire structure. Hence only the external force **P** and the reaction forces from the constraints are shown. The second example shows the typical reaction force of the cantilever beam that bears the shear force, the axial force, and the bending moment. The third example shows the reaction force of the smooth surface contact, which is always perpendicular to the contact surface. The last example reveals the effectiveness of hiding unwanted unknowns, the cable tension in this case, inside the boundary of the FBD.

Example 3.1 ([1], Prob. 3/A,B) Complete the FBDs given in fig. 3.4 and 3.5.

Solution: By following the guideline given above, the complete FBDs as shown in fig. 3.6 and 3.7 result.

3.3 2-D Equilibrium Conditions

From the Newton's second law, a body is in equilibrium if all forces and moments applied to it are in balance. For 2-D problems, the above conditions can be written in formula as

$$\sum F_x = 0 \tag{3.3}$$

$$\sum F_y = 0 \tag{3.4}$$

$$\sum M_O = 0 \tag{3.5}$$

After we have finished writing the FBD, the equilibrium condition can then be applied. FBD will give the answer of the left hand side of the equations, which

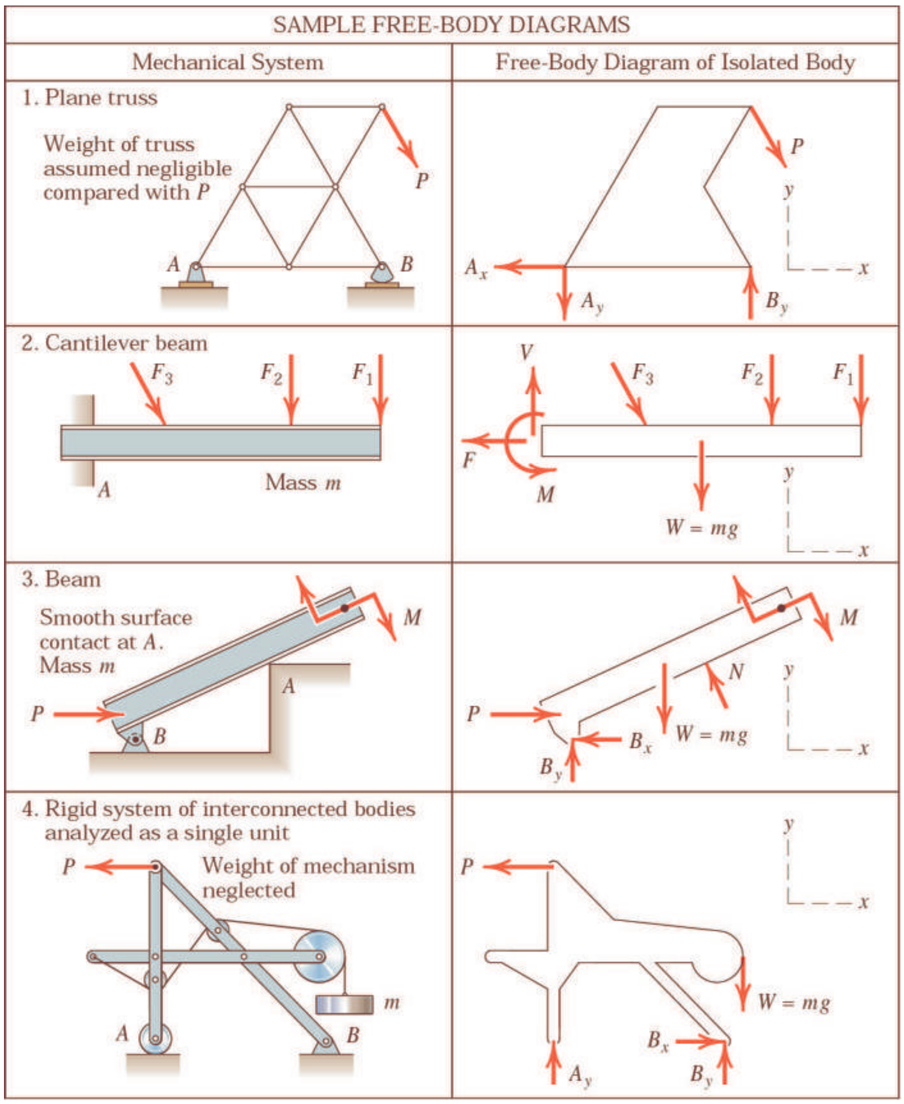


Figure 3.3: Examples of drawing the free body diagram ([1], pp. 115)

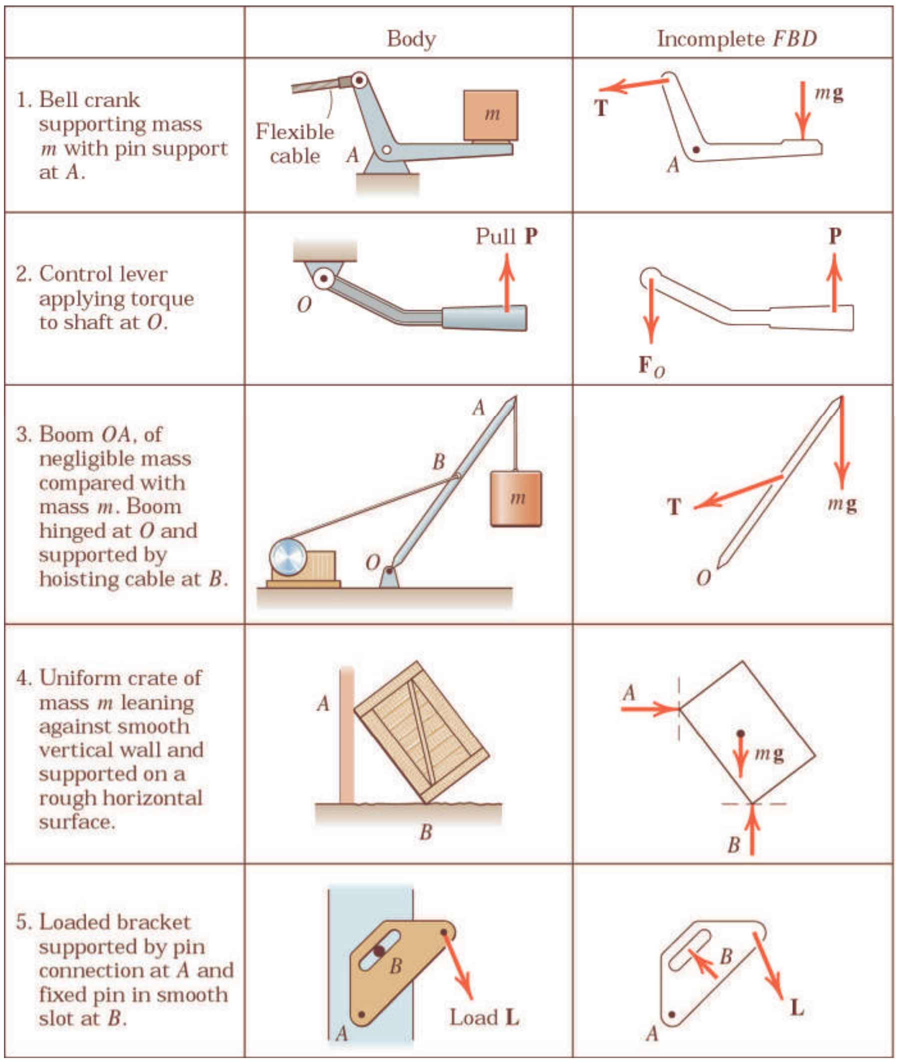


Figure 3.4: Example 3.1 ([1], pp. 118)

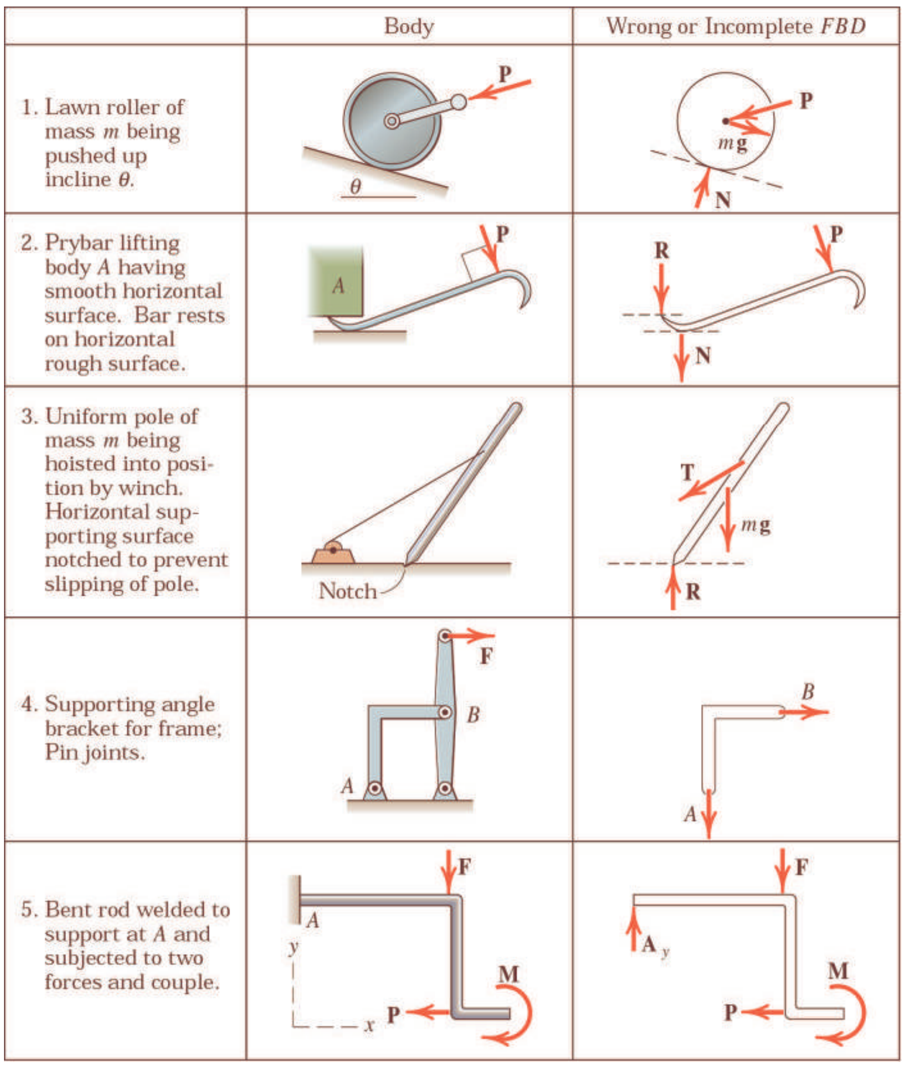


Figure 3.5: Example 3.1 (continued) ([1], pp. 119)

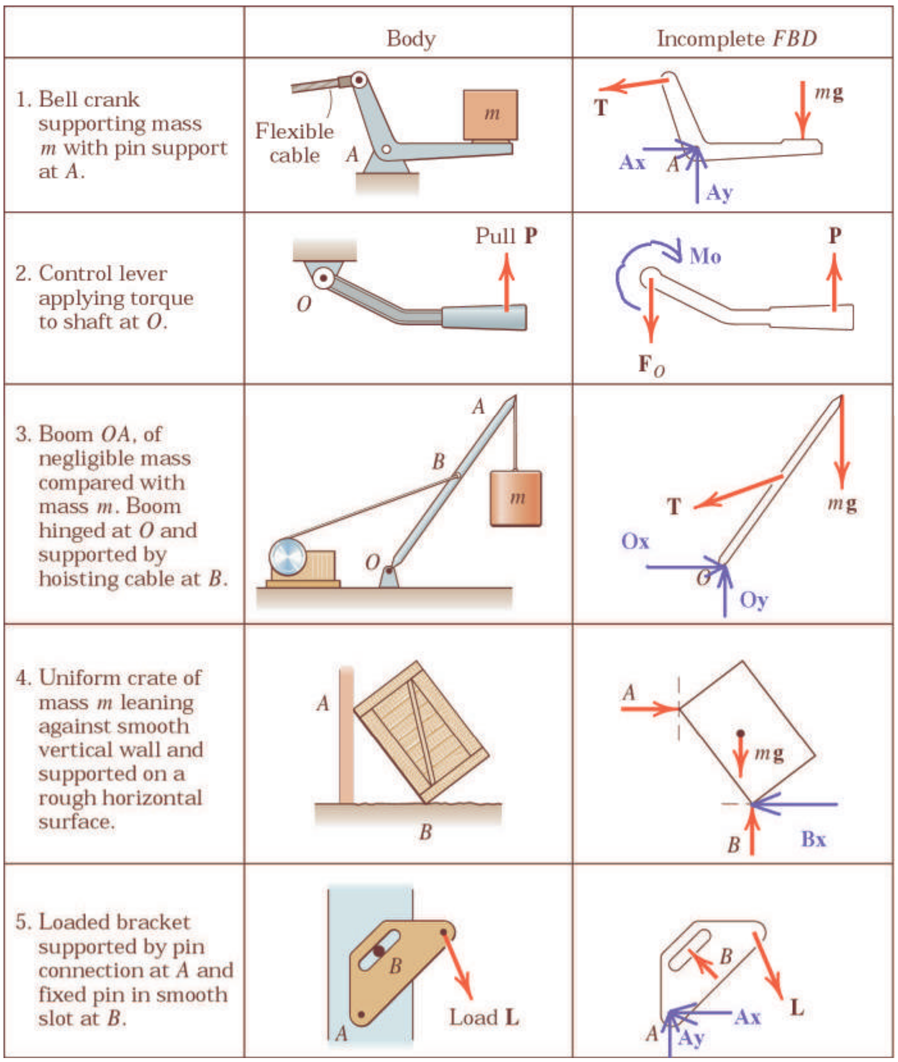


Figure 3.6: Solution to example 3.1

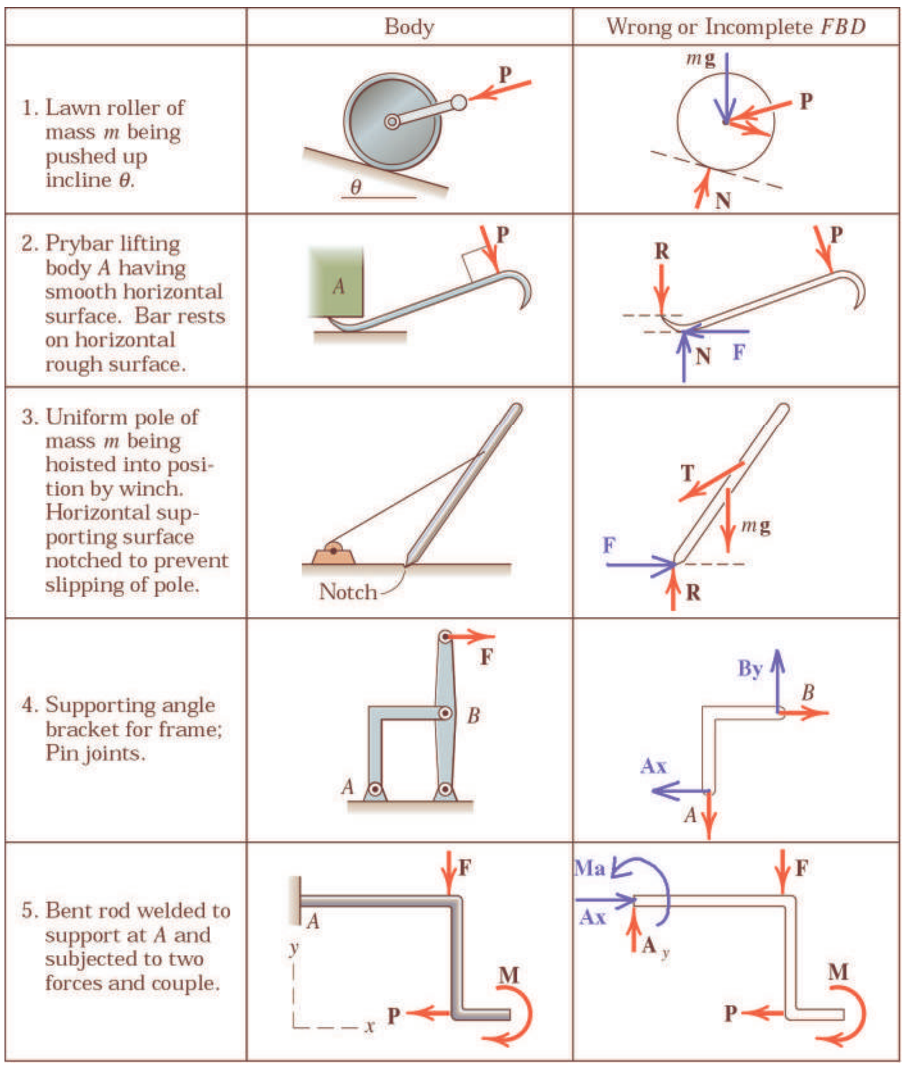


Figure 3.7: Solution to example 3.1 (continued)

are the resultant of the force system acting on the body of interest. Since the object is in equilibrium, the resultant must be nulled and hence the zero value on the right hand side. There are some things needed to be mentioned in applying the equations.

1. The x-y coordinate system and the moment point O can be chosen arbitrarily.

2. Complete equilibrium in 2-D motion must satisfy all three equations. However, they are independent to each other. That is, equilibrium may only be satisfied in some generalized coordinates.

3. System in equilibrium may stay still or *move* with constant velocity. In both cases, the acceleration is zero.

At this point, the reader is armed with enough information to start practicing the equilibrium problems. However it is helpful to know the characteristics of some special cases of equilibrium problems. If we classify the problem according to the force system, the following cases of equilibrium in two dimension, fig. 3.8, may result.

If the force system is collinear, only one equilibrium equation $\sum F_x = 0$ is effective. The other equations (perpendicular force and moment equations) are satisfied automatically and hence contribute nothing in solving the problems. For the concurrent force system, the moment at the point of concurrency is always satisfied. Therefore only the force equilibrium equations are usable. In the case of parallel forces, the force in the perpendicular direciton is null. Hence only the force equilibrium equation in the parallel direction and the moment equation are active. Note that in general case of force system, all three equilibrium equations are valid.

Two-force member is a body under the action of two force only. The equilibrium condition of the two-force member requires the acting forces be equal, opposite, and collinear. Only one equation of force along its direction is then effective. One common assumption of the two-force member is that the weight of the member is negligible. Sometimes it is the trade-off between the simplification in solving the equilibrium problem and the accuracy of the answer. Figure 3.9 depicts some two-force members.

Three-force member is a body under the action of three forces only. The equilibrium condition of the three-force member requires the lines of action of the three forces be concurrent. The only exception is when the three forces are parallel. In this case, two equations of forces are effective. The moment equation is satisfied automatically. The equivalent requirement is the closure of the polygon of forces. See fig. 3.10. In many cases, the force system may be

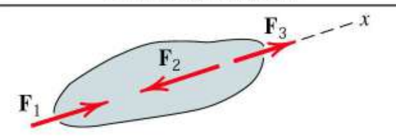
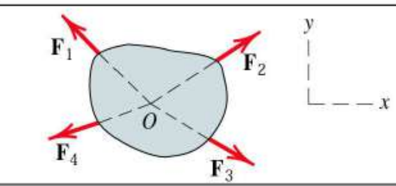
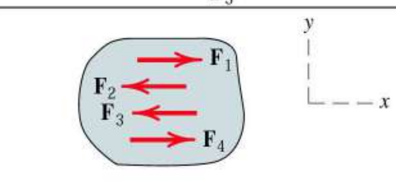
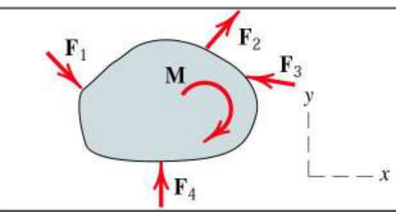
CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\Sigma M_z = 0$ $\Sigma F_y = 0$

Figure 3.8: Special cases of equilibrium in two dimension ([1], pp. 122)

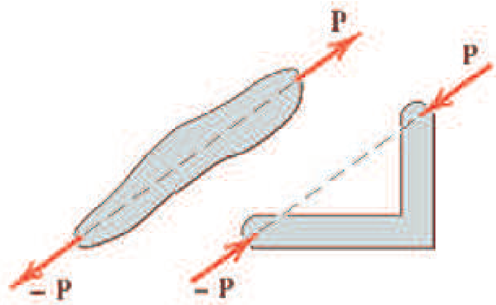


Figure 3.9: Two-force member ([1], pp. 122)

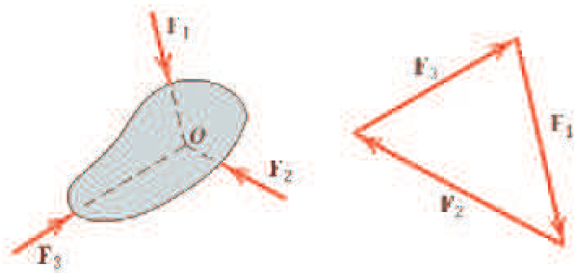


Figure 3.10: Three-force member ([1], pp. 123)

reduced to the three-force member by successive addition of the known forces.

Equation 3.3, 3.4, and 3.5 are not the only set of equations for equilibrium. Here we represent another two sets of equations which also guarantee the equilibrium of an object. The first set of equilibrium conditions are

$$\sum F_x = 0 \tag{3.6}$$

$$\sum M_A = 0 \tag{3.7}$$

$$\sum M_B = 0 \tag{3.8}$$

where $\overline{AB}$ must not be perpendicular to the x-direction. See fig. 3.11 for the equation setup. $\sum F_x = 0$ forces the reaction to be only in the perpendicular direction. $\sum M_A = 0$ requires that reaction to pass through A, and so do $\sum M_B = 0$ constraints the reaction to pass through B. However all three requirements will be satisfied only if the resultant is zero, otherwise the moment equations will make the nonzero resultant break the force equation.

Another set of equilibrium conditions are

$$\sum M_A = 0 \tag{3.9}$$

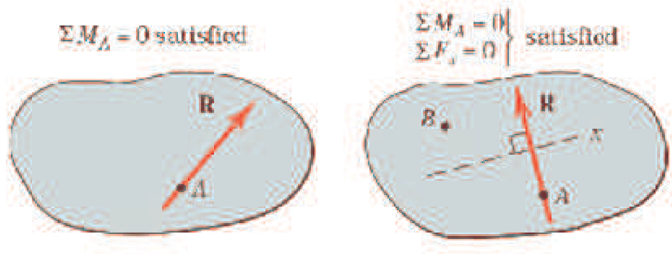


Figure 3.11: An alternative equilibrium conditions ([1], pp. 123)

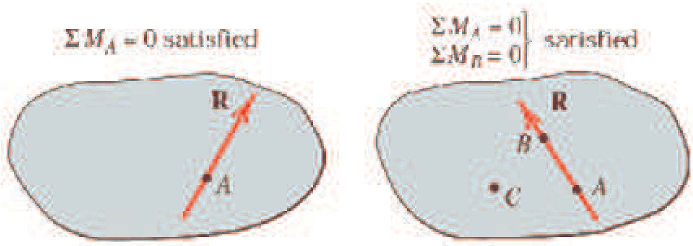


Figure 3.12: Another alternative equilibrium conditions ([1], pp. 123)

$$\sum M_B = 0 \tag{3.10}$$

$$\sum M_C = 0 \tag{3.11}$$

where A, B, and C are not on the same straight line. See fig. 3.12 for the equation setup. $\sum M_A = 0$ and $\sum M_B = 0$ force the reaction to pass through A and B. The equation $\sum M_C = 0$ does the same thing. However all three requirements will be satisfied only if the resultant is zero, otherwise the last moment equation will make the nonzero resultant break the first two equations.

Constraints and Statical Determinacy The equilibrium equation may not always solve *all* unknowns in the problem. This is because the equilibrium condition do not provide enough equations. Simply put, if the number of unknowns (including geometrical variables) is greater than the number of equations, then we cannot solve it. This is because the system has more constraints than necessary to maintain the equilibrium. This is called *statically indeterminate system*. Extra equations obtained from force-deformation material properties must also be applied to solve for the redundant constraints. If the number of unknown is equal or less than the number of equilibrium equations, the system is *statically determinate*. With some wrong installation of the supports, the number of active constraints may be reduced unintentionally. See case (b) and (c) of fig. 3.13.

Before giving some examples, let us review the guideline in solving the equi-

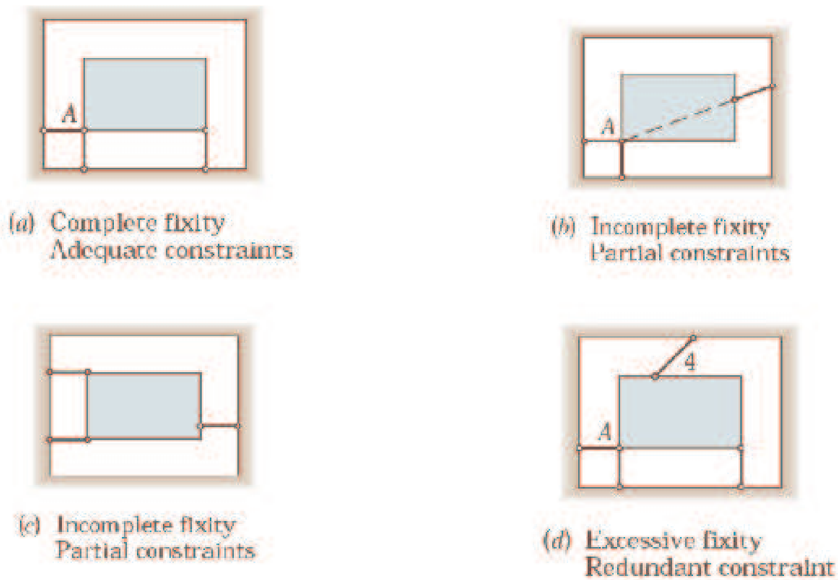


Figure 3.13: Two Dimensional Constraints and Statical Determinacy ([1], pp. 125)

librium problems.

1. List known / unknown quantities. Then count the number of unknowns and the number of available independent equations. If the number of unknowns is greater than the number of equations, the problem cannot be solved solely by the equilibrium conditions.
2. *Determine the isolated system and draw the FBD.*
3. Assign a convenient set of coordinate systems. Choose suitable moment centers for calculation.
4. Write down the governing equations, e.g. $[\sum M_O = 0]$, before the calculation.
5. Choose the suitable method in solving the problem; whether it be scalar, vector, or geometric approach.

Example 3.2 ([1], Prob. 3/32) In a procedure to evaluate the strength of the triceps muscle, a person pushes down on a load cell with the palm of his hand as indicated in the figure. If the load-cell reading is 160 N, determine the vertical tensile force **F** generated by the triceps muscle. The mass of the lower arm is 1.5 kg with mass center at G. State any assumptions.

Solution: Let us choose the system to be the lower arm. Consequently,

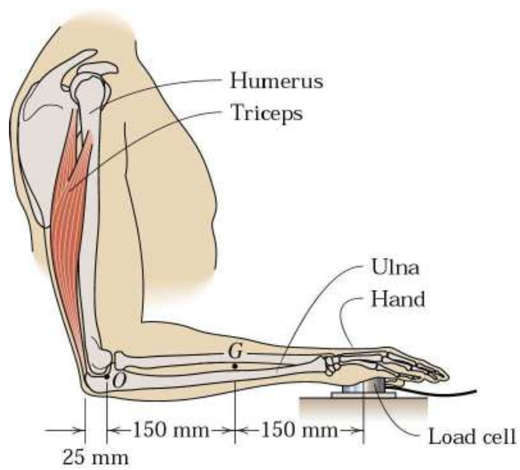


Figure 3.14: Example 3.2 ([1], pp. 137)

the forces acting on it are the pushing force from the load-cell, the mass of the arm, the tensile force by the triceps muscle, and the compressive force by the humerus bone. The FBD of the system is shown in fig. 3.15. From the figure, there are two unknowns. Hence two equations are required, which are one moment and one force equations. First let us take the moment about O to eliminate the compressive force.

$$[\sum M_O = 0] \qquad -T \times 25 - 1.5g \times 150 + 160 \times 300 = 0$$

$$T = 1832 \text{ N}$$

Force equation is then used to determine the tensile force produced by the triceps, i.e.,

$$[\sum F_y = 0] \qquad T - C - 1.5g + 160 = 0$$

$$C = 1977 \text{ N}$$

Example 3.3 ([1], Prob. 3/39) The exercise machine consists of a lightweight cart which is mounted on small rollers so that it is free to move along the inclined ramp. Two cables are attached to the cart – one for each hand. If the hands are together so that the cables are parallel and if each cable lies essentially in a vertical plane, determine the force **P** which each hand must exert on its cable in order to maintain an equilibrium position. The mass of the person is 70 kg, the ramp angle is 15°, and the angle β is 18°. In addition, calculate the force **R**

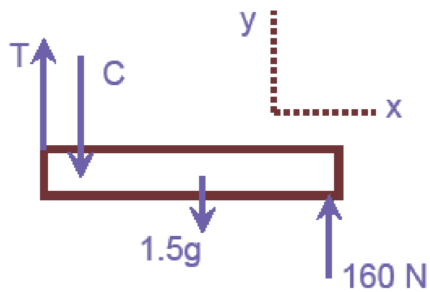


Figure 3.15: Solution to example 3.2

Figure 3.16: Example 3.3 ([1], pp. 139)

which the ramp exerts on the cart.

Solution: Assume the cart is lightweight and the rail friction is negligible. Then we select the machine and the man as our system and draw the FBD as shown in fig. 3.17. It is seen that there are two unknowns which are solvable using two force equations.

$$[\sum F_{x'} = 0] \qquad 70g \sin 15 - T \cos 9 = 0 \qquad T = 179.9 \text{ N}$$

$$[\sum F_{y'} = 0] \qquad R - 70g \cos 15 - T \sin 9 = 0 \qquad R = 691 \text{ N}$$

To determine the force exerted by each hand, we choose the pulley and cable as our system of interest and draw its FBD, as depicted in fig. 3.18. Only the force exerted by each hand, **P**, is the unknown. The force along the line of action of **T** must be null which means

$$[\sum F_{x'} = 0] \qquad -T + 4P \cos 9 = 0 \qquad P = 45.5 \text{ N}$$

Example 3.4 ([2], Prob. 3/29) A uniform ring of mass m and radius r carries

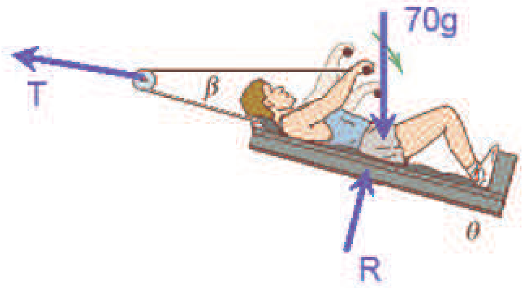


Figure 3.17: Solution to example 3.3

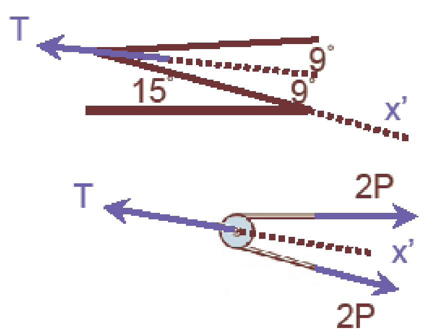


Figure 3.18: Solution to example 3.3 (continued)

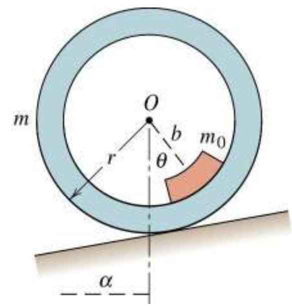


Figure 3.19: Example 3.4 ([2], pp. 131)

an eccentric mass m_o at a radius b and is in an equilibrium position on the incline, which makes an angle α with the horizontal. If the contacting surfaces are rough enough to prevent slipping, write the expression for the angle θ which defines the equilibrium position.

Solution: Write the FBD of the system which consists of the ring and the eccentric mass. The force acting on the system are the weights, the friction force, and the normal force. Here the friction is not negligible and indeed makes the system be in equilibrium. We have three unknowns, which are F, N, and θ. However, the problem only asks us for the angle θ.

First we should take the moment equation about O to eliminate the unknown N.

$$[\sum M_O = 0] \qquad Fr - m_o gb \sin\theta = 0 \qquad F = \frac{m_o gb \sin\theta}{r}$$

Then we use the force equilibrium equation in the x'-direction and substitute the expression of F. Note that the normal force does not appear in the equation.

$$[\sum F_{x'} = 0] \qquad F - (m_o + m)\, g \sin\alpha = 0 \qquad \theta = \text{asin}\left[\frac{r}{b}\left(1 + \frac{m}{m_o}\right)\sin\alpha\right]$$

Example 3.5 ([1], Prob. 3/43) The hook wrench or pin spanner is used to turn shafts and collars. If a moment of 80 Nm is required to turn the 200 mm diameter collar about its center O under the action of the applied force **P**, determine the contact force **R** on the smooth surface at A. Engagement of the pin at B may be considered to occur at the periphery of the collar.

Solution: First select the system of the wrench and the collar as shown in fig. 3.22. From the FBD, the applied force **P** must be

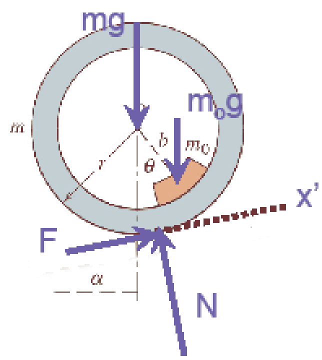


Figure 3.20: Solution to example 3.4

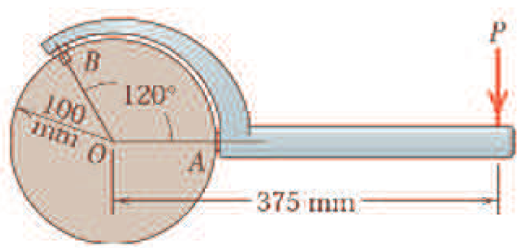


Figure 3.21: Example 3.5 ([1], pp. 140)

$[\sum M_O = 0]$ $\quad 80 - P \times 0.375 = 0 \qquad P = 213.3 \;\; \text{N}$

Now we must select only the spanner as our system which will reveal the reaction force at A as shown in fig. 3.23. Note that this is a three-force member. The moment equilibrium at B suppress the reaction $\mathbf{R}$ to appear in the equation.

$[\sum M_B = 0]$ $\quad N_A \times 0.1 \sin 60 - P \times (0.375 + 0.1 \cos 60) = 0 \qquad N_A = 1047 \;\; \text{N}$

Example 3.6 ([1]) The small crane is mounted on one side of the bed of a pickup truck. For the position $\theta = 40°$, determine the magnitude of the force supported by the pin at O and the oil pressure p against the 50 mm-diameter piston of the hydraulic cylinder BC.

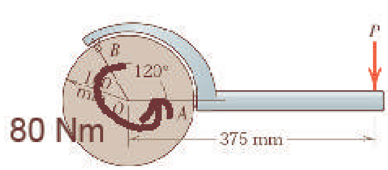


Figure 3.22: Solution to example 3.5

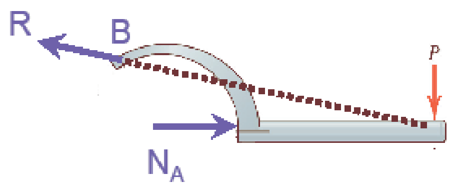


Figure 3.23: Solution to example 3.5 (continued)

Solution: To determine such forces, we select the linkage ACO as our system and draw its FBD. Note that this is a three-force member. Hence three reaction forces must meet at a common point, which helps in drawing the correct forces shown in fig. 3.25. However, before setting up the equilibrium equations, let us consider the geometry at $\overline{BCDO}$ to determine the moment arm of the hydraulic force. Referring to fig. 3.26,

$$\alpha = \text{atan}\left(\frac{360 + 340\sin 40 - 110\cos 40}{340\cos 40 + 110\sin 40}\right) = 56.2^\circ$$

The perpendicular distance from $\overline{BC}$ to O is then

$$d = 360\cos\alpha = 200 \quad \text{mm}$$

Now we are ready to apply the equilibrium conditions to the system. Start with the moment equation at O to eliminate unknown reaction at O.

$$[\sum M_O = 0] \qquad 120g \times (785 + 340)\cos 40 - C \times d = 0 \qquad C = 5063 \quad \text{N}$$

Therefore the oil pressure of the hydraulic cylinder is

$$p = \frac{C}{\pi r^2} = 2.58 \quad \text{MPa}$$

The force equations will be used in determining the reaction supported by the pin at O.

$$[\sum F_x = 0] \qquad O_x - C\cos\alpha = 0 \qquad O_x = 2820 \quad \text{N}$$

$$[\sum F_y = 0] \qquad -O_y - 120g + C\sin\alpha = 0 \qquad O_y = 3030 \quad \text{N}$$

$$O = \sqrt{O_x^2 + O_y^2} = 4140 \quad \text{N}$$

Example 3.7 ([2], Prob. 3/48) The rubber-tired tractor shown has a mass of 13.5 Mg with the center of mass at G and is used for pushing or pulling

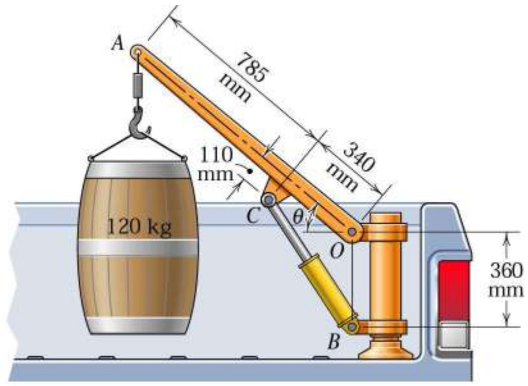


Figure 3.24: Example 3.6 ([1])

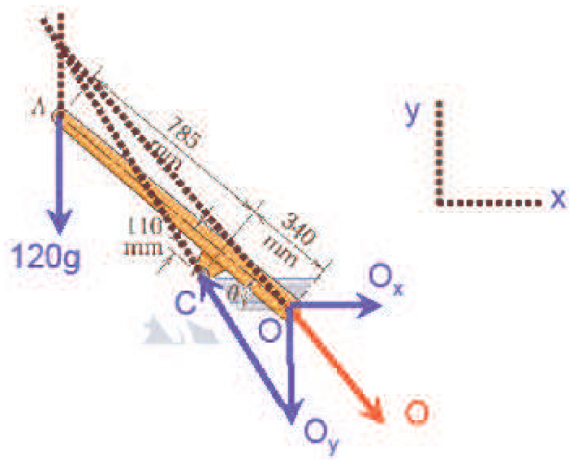


Figure 3.25: Solution to example 3.6

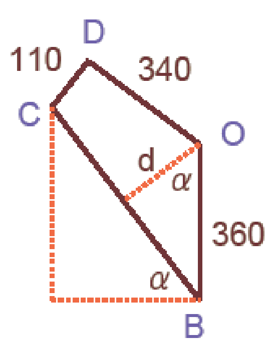


Figure 3.26: Solution to example 3.6 (continued)

heavy loads. Determine the load **P** which the tractor can pull at a constant speed of 5 km/h up the 15-percent grade if the driving force exerted by the ground on each of its four wheels is 80 percent of the normal force under that wheel. Also find the total normal reaction $\mathbf{N}_B$ under the rear pair of wheels at B.

Solution: The inclined coordinate system is suitable for this problem. We draw the FBD of the truck as our system. The forces are its weight, the pulling load **P**, the friction force, and the normal force on the wheels. Note that the reaction at the front and rear wheels are different due to the incline and the center of gravity is not at the midway between the wheels.

The problem states the driving force is 80 percent of the normal force of that wheel. Hence there is only one unknown for each wheel. Totally there are 3 unknowns; the normal forces on the wheels and the pulling force. Let us apply the force equilibrium equations first.

$$[\sum F_{x'} = 0] \qquad P - 0.8N_A - 0.8N_B + 13500g \times \tfrac{15}{\sqrt{15^2+100^2}} = 0$$

$$[\sum F_{y'} = 0] \qquad N_A + N_B - 13500g \times \tfrac{100}{\sqrt{15^2+100^2}} = 0$$

We see that there are three unknowns in two equations. Another equation from the moment equilibrium must be solved simultaneously. Let us choose to take the moment about A.

$$[\sum M_A = 0] \qquad N_B \times 1.8 - P \times 0.6 - 13500g \times \tfrac{100}{\sqrt{15^2+100^2}} \times 1.2$$
$$-13500g \times \tfrac{15}{\sqrt{15^2+100^2}} \times 0.825 = 0$$

Three unknowns are then determined as

$$N_A = 6.3 \ \ \text{kN}, \qquad N_B = 124.7 \ \ \text{kN}, \qquad P = 85.1 \ \ \text{kN}$$

Alternatively, we may use a different set of equilibrium equations

$$\left[\sum M_A = 0\right] \quad \left[\sum M_B = 0\right] \quad \left[\sum F_{x'} = 0\right]$$

in solving for the unknowns.

Example 3.8 ([1], Prob. 3/59) Pulley A delivers a steady torque (moment) of 100 Nm to a pump through its shaft at C. The tension in the lower side of the belt is 600 N. The driving motor B has a mass of 100 kg and rotates clockwise. Determine the magnitude R of the force on the supporting pin at O.

Solution: To determine the reaction at O, we must know the reaction at the spring support and the upper side tension. This leads us to draw the FBD

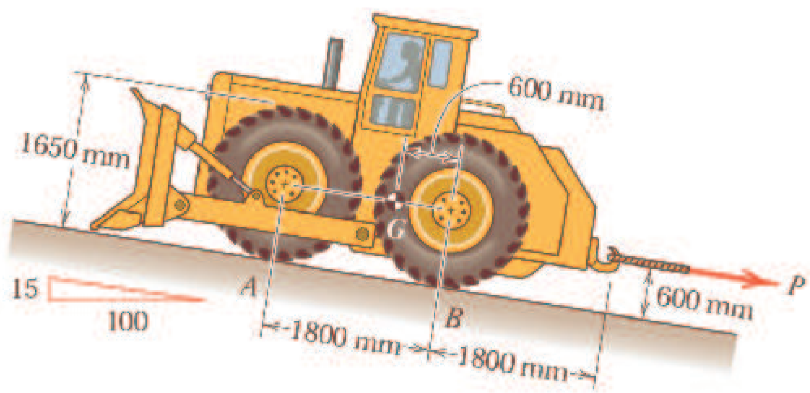


Figure 3.27: Example 3.7 ([2], pp. 136)

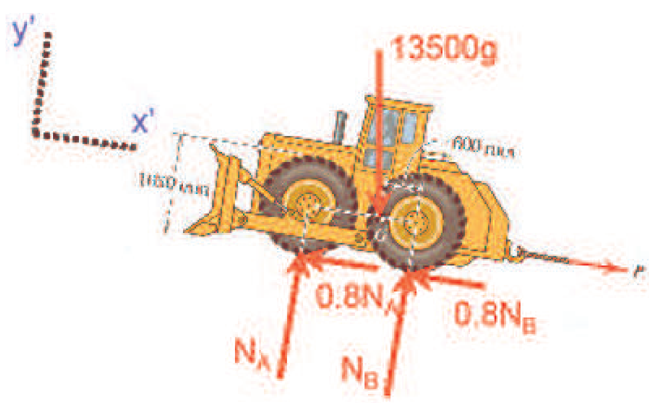


Figure 3.28: Solution to example 3.7

of the pulley A. Since the pulley A delivers 100 Nm torque to a pump, there must be the moment reaction resisting the clockwise rotation of it. See fig. 3.30. The moment equilibrium is then used to determine the tension force, T.

$$[\sum M_C = 0] \qquad (600 - T) \times 0.225 - 100 = 0$$

$$T = 155.6 \;\; \text{N}$$

Next we draw the FBD of the motor, fig. 3.31, of which the supporting forces **O** and **P** are revealed. Determining three unknowns are relatively straightforward.

$$[\sum M_D = 0] \qquad O_y \times 0.25 - 600 \times (0.2 - 0.075) - 100g \times 0.125$$

$$-T \times 0.075 - T\cos 30 \times 0.2 + T \sin 30 \times 0.125 = 0$$

$$O_y = 906 \;\; \text{N}$$

$$[\sum F_x = 0] \qquad T\cos 30 + 600 - O_x = 0 \qquad O_x = 734.7 \;\; \text{N}$$

$$O = \sqrt{O_x^2 + O_y^2} = 1.17 \;\; \text{kN}$$

$$[\sum F_y = 0] \qquad T\sin 30 - 100g - P + O_y = 0 \qquad P = 2.8 \;\; \text{N}$$

Note the tension in the rod at the left leg results from the compressed spring to resist the rotation of the motor stator.

Example 3.9 ([2], Prob. 3/52) When setting the anchor so that it will dig into the sandy bottom, the engine of the 40 Mg cruiser with the center of gravity at G is run in reverse to produce a horizontal thrust **T** of 2 kN. If the anchor chain makes an angle of 60° with the horizontal, determine the forward shift b of the center of buoyancy from its position when the boat is floating free. The center of buoyancy is the point through which the resultant of the buoyant force passes.

Solution: When the boat is free floating (no thrust or tension), the buoyancy force is equal to the weight and acting at the C.G. At the times the boat moves backward, the anchor chain is pulled against the thrust force. This changes the buoyancy force, both in magnitude and point of application, to maintain the equilibrium. Figure 3.33 is the FBD of this system. Apply the equilibrium equations to solve for three unknowns; A, B, and b.

$$[\sum F_x = 0] \qquad A\cos 60 - 2000 = 0 \qquad A = 4 \;\; \text{kN}$$

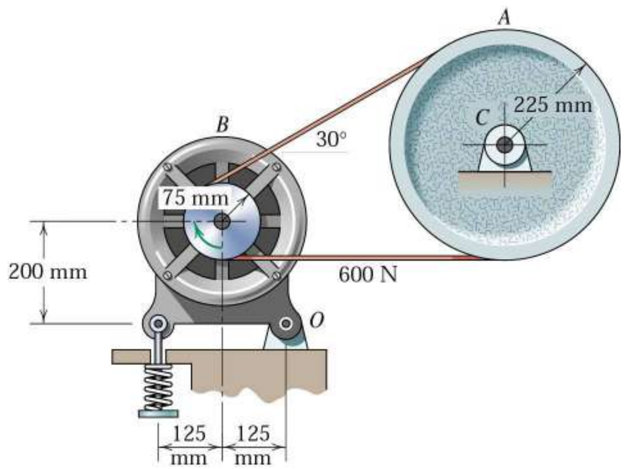


Figure 3.29: Example 3.8 ([1], pp. 144)

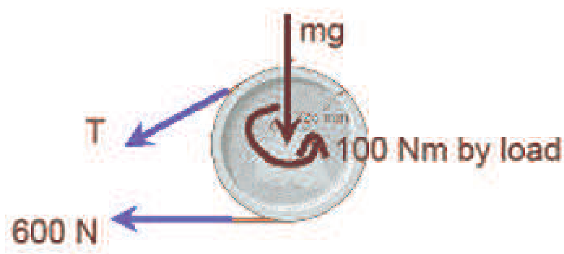


Figure 3.30: Solution to example 3.8

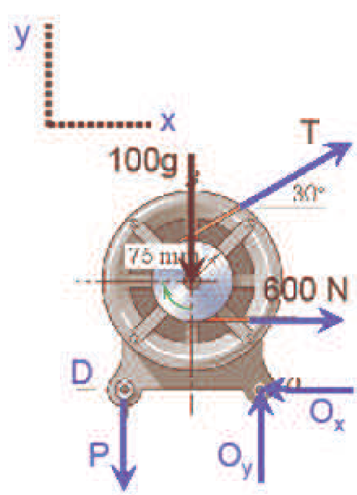


Figure 3.31: Solution to example 3.8 (continued)

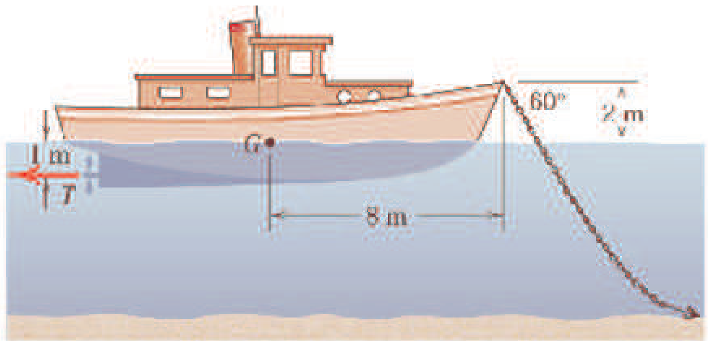


Figure 3.32: Example 3.9 ([2], pp. 137)

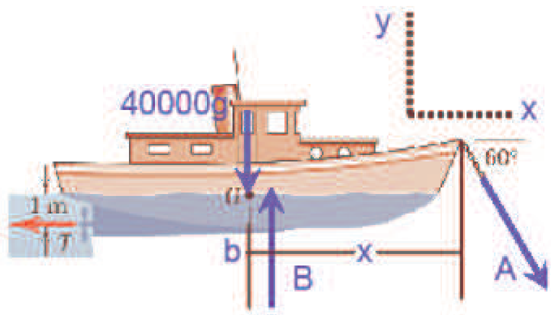


Figure 3.33: Solution to example 3.9

$$[\sum F_y = 0] \qquad B - 40000g - A\sin 60 = 0 \qquad B = 395,864 \ \text{ N}$$

$$[\sum M_A = 0] \qquad 40000g \times 8 - 2000 \times 3 - Bx = 0 \qquad x = 7.915 \ \text{ m}$$

$$b = 8 - x = 85.2 \ \text{ mm}$$

Example 3.10 ([2], Prob. 3/63) A special jig for turning large concrete pipe sections (shown dotted) consists of an 80 Mg sector mounted on a line of rollers at A and a line of rollers at B. One of the rollers at B is a gear which meshes with a ring of gear teeth on the sector so as to turn the sector about its geometric center O. When $\alpha = 0$, a counterclockwise torque of 2460 Nm must be applied to the gear at B to keep the assembly from rotating. When $\alpha = 30°$, a clockwise torque of 4680 Nm is required to prevent rotation. Locate the mass center G of the jig by calculating r and θ.

Solution: The problem gives two different postures that are in equilibrium and ask for the locating the C.G. by the parameters r and θ. Therefore two equations coming from each posture of equilibrium must be constituted and solved. For the case when $\alpha = 0$, a CCW torque is applied to the gear B. Therefore the reaction force at the meshing teeths with the jig must be as shown

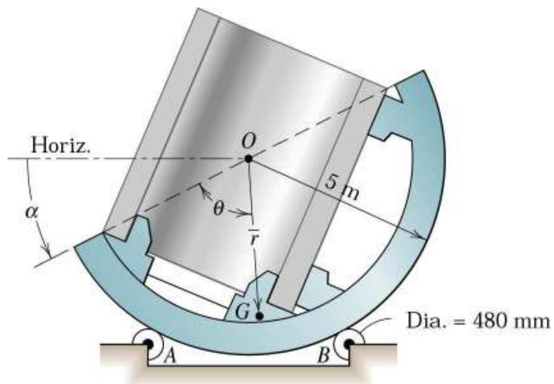


Figure 3.34: Example 3.10 ([2], pp. 141)

in fig. 3.35. For the case of $\alpha = 30°$, a CW torque is applied instead. Hence the reaction force at the meshing teeths is as shown in fig. 3.36. In each case, the tangential component of the reaction force is determined by the moment equilibrium equation:

$$[\sum M_B = 0] \qquad \alpha = 0° : \quad 2460 - F_1 \times 0.24 = 0 \qquad F_1 = 10250 \;\; \text{N}$$

$$\alpha = 30° : -4680 + F_2 \times 0.24 = 0 \qquad F_2 = 19500 \;\; \text{N}$$

The reaction is imparted to the jig, shown in fig. 3.37, along with the one from the rollers A. To avoid determining the unnecessary unknowns, N_A and N_B, we should use the moment equation taken at the center point O. For each case, we have

$$[\sum M_O = 0] \qquad \alpha = 0° : \quad 80000g \times r\cos\theta - 10250 \times 5 = 0$$

$$\alpha = 30° : \quad -80000g \times r\cos(180 - 30 - \theta) + 19500 \times 5 = 0$$

Solving the above equations simultaneously, the location of the C.G. is determined.

$$r = 367 \;\; \text{mm} \qquad \theta = 79.8°$$

Figure 3.35: Solution to example 3.10

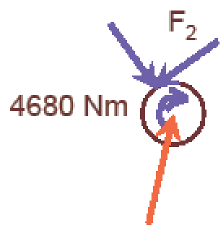


Figure 3.36: Solution to example 3.10 (continued)

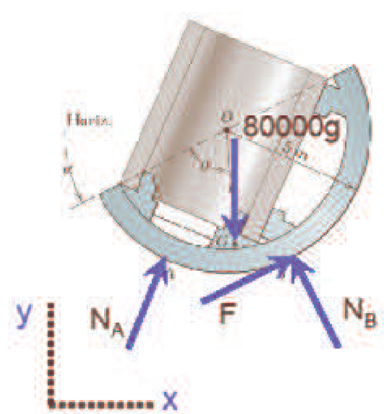


Figure 3.37: Solution to example 3.10 (continued)

3.4 3-D Equilibrium Conditions

From the Newton's second law, a body is in equilibrium if all forces and moments applied to it are in balance. For 3-D problems, the above conditions can be written in formula as

$$\sum F_x = 0 \tag{3.12}$$

$$\sum F_y = 0 \tag{3.13}$$

$$\sum F_z = 0 \tag{3.14}$$

$$\sum M_{O_x} = 0 \tag{3.15}$$

$$\sum M_{O_y} = 0 \tag{3.16}$$

$$\sum M_{O_z} = 0 \tag{3.17}$$

After we have finished writing the FBD, the equilibrium condition can then be applied. FBD will give the answer of the left hand side of the equations, which are the resultant of the force system acting on the body of interest. Since the object is in equilibrium, the resultant must be nulled and hence the zero value on the right hand side. There are some things needed to be mentioned in applying the equations.

1. The x-y-z coordinate system and the moment point O can be chosen arbitrarily.

2. Complete equilibrium in 3-D motion must satisfy all six equations. However, they are independent to each other. That is, equilibrium may only be satisfied in some generalized coordinates.

3. System in equilibrium may stay still or *move* with constant velocity. In both cases, the acceleration is zero.

Figure 3.38 shows common examples of modeling the action of forces in three dimensional problems. Beware these are not the FBDs. Only the specific action forces, after the surroundings have been removed, are shown. Here are some explanation of each particular case.

1. Reaction force of the member in contact with the smooth surface, or ball-supported member, is normal to the surface and directed toward the member. It is usually denoted by **N**.

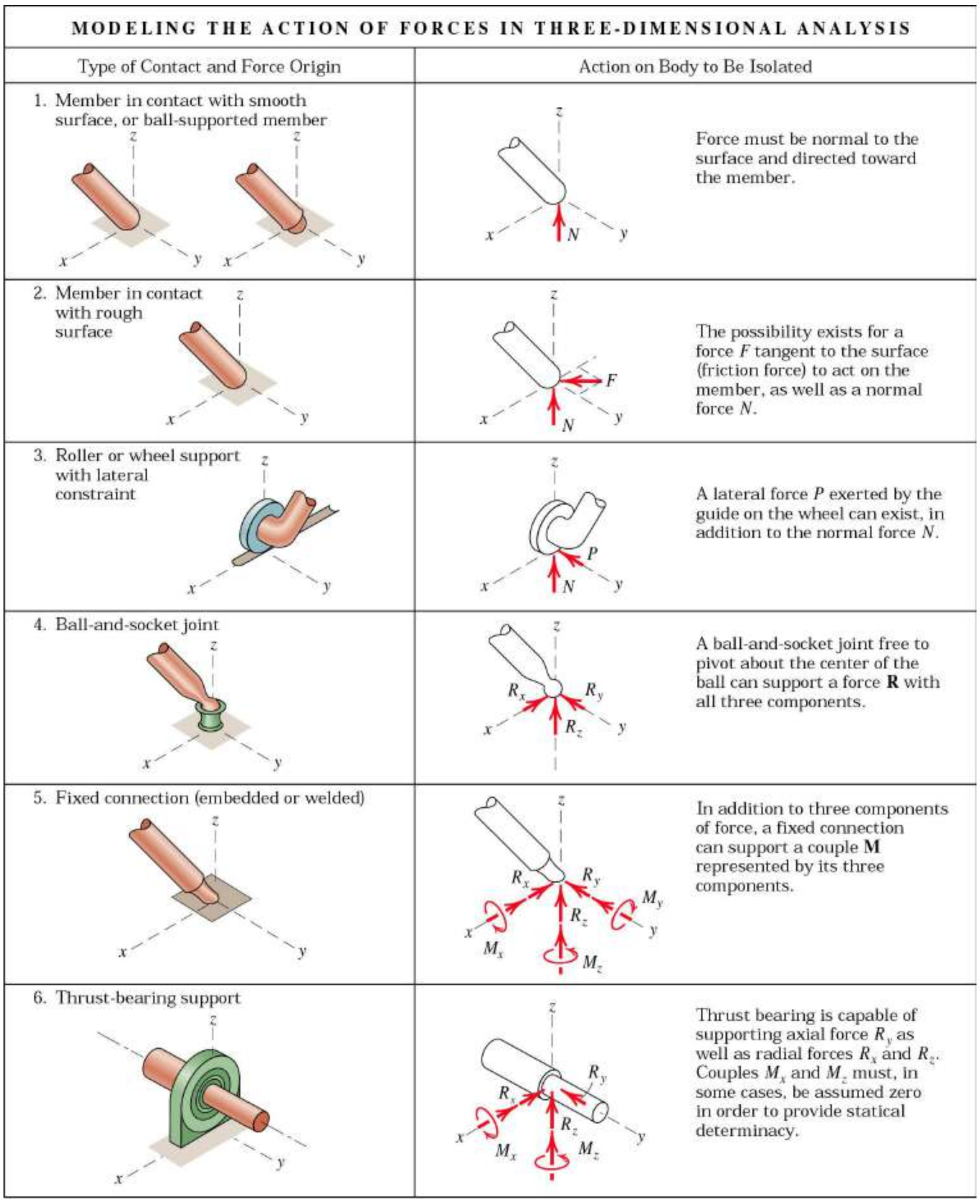


Figure 3.38: Common action of forces in three dimensional analysis ([1], pp. 147)

2. Reaction force of the member in contact with the rough surface has both the normal and tangential components. To determine these forces, we first seek for the contact plane which is tangent to both contact surfaces. Normal force, **N**, acts in the direction normal to the plane while tangential force, **F**, lies in the plane. Tangential force is commonly recognized as the friction force.

3. Usually there will be additional reaction force corresponding to the additional constraint. In the example, the lateral force **P** is introduced as the rail also prevents the wheel from the sideway motion, in addition to the normal force **N**.

4. The ball and socket joint constraints the point on each linkage to always be together. This requires the joint to support a force **R**.

5. If the joint is welded or completely embedded, two linkages cannot be moved relative to each other. Therefore the additional supporting moments must be ensured to prevent the rotational motion.

6. The thrust bearing support prevents the shaft from moving and rotating in all direction except the rotational motion about the shaft axis. Hence three dimensional reaction forces, R_x, R_y, and R_z, must be supplied. In addition, the resisting moment in x and z- direction must be provided as well.

At this point, the reader is armed with enough information to start practicing the equilibrium problems. However it is helpful to know the characteristics of some special cases of equilibrium problems. If we classify the problem according to the force system, the following cases of equilibrium in three dimension, fig. 3.39, may result.

For the concurrent force system, the moment at the point of concurrency is always satisfied. Therefore only the force equilibrium equations are usable. If the force system has all forces being concurrent with a line, i.e. all forces intersecting a common line, then the moment of the forces about that line is automatically null. Therefore only five equilibrium equations are effective. In the case of parallel forces, the force in the perpendicular direction is null. Hence only the force equilibrium equation in the parallel direction and the moment equations about the lines in the plane perpendicular to the parallel forces are active. Note that in general case of force system, all six equilibrium equations are valid.

Constraints and Statical Determinacy The equilibrium equation may not always solve *all* unknowns in the problem. This is because the equilibrium condition do not provide enough equations. Simply put, if the number of unknowns (including geometrical variables) is greater than the number of equations,

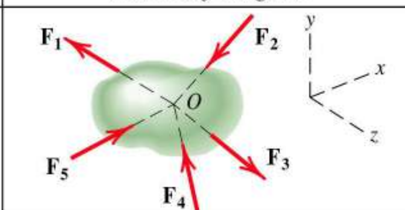
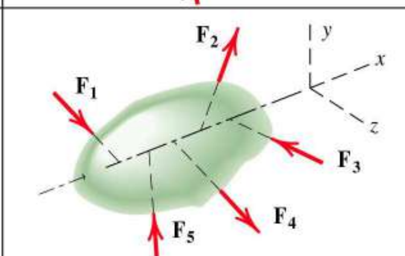
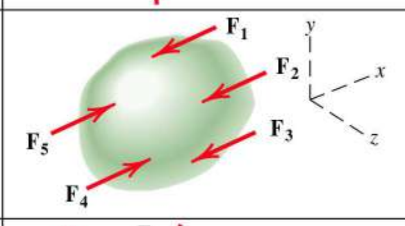
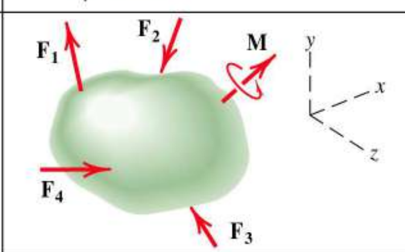
CATEGORIES OF EQUILIBRIUM IN THREE DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$
2. Concurrent with a line		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma F_y = 0$ $\Sigma M_z = 0$ $\Sigma F_z = 0$
3. Parallel		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\Sigma M_x = 0$ $\Sigma F_y = 0$ $\Sigma M_y = 0$ $\Sigma F_z = 0$ $\Sigma M_z = 0$

Figure 3.39: Special cases of equilibrium in three dimension ([1], pp. 148)

then we cannot solve it. This is because the system has more constraints than necessary to maintain the equilibrium. This is called *statically indeterminate system*. Extra equations obtained from force-deformation material properties must also be applied to solve for the redundant constraints. If the number of unknown is equal or less than the number of equilibrium equations, the system is *statically determinate*. With some wrong installation of the supports, the number of active constraints may be reduced unintentionally. See case (b) and (c) of fig. 3.40.

Example 3.11 ([2], Prob. 3/71) The light right angle boom which supports the 400 kg cylinder is supported by three cables and a ball-and-socket joint at O attached to the vertical x-y surface. Determine the reactions at O and the cable tensions.

Solution: The FBD of this problem is shown in fig. 3.42. There are six unknowns; three reaction force components at O and three cable tensions. We can simply set up three force equilibrium equations and three moment equilibrium equations about any point. However solving the resulting equations may be intractable. A little thought before writing down the solution can be helpful. For this problem, the moment equilibrium about line $\overline{OB}$ suppresses five unknowns to show up. Only T_{AC} will appear in the equation, which is easy to solve. Examining the FBD in this manner helps avoid solving simultaneous equations unnecessarily.

First, let us determine the pertaining unit vectors:

$$\begin{aligned} \mathbf{n}_{\mathrm{AC}} &= -0.408\mathbf{i} + 0.408\mathbf{j} - 0.816\mathbf{k}, & \mathbf{n}_{\mathrm{BD}} &= 0.707\mathbf{j} - 0.707\mathbf{k} \\ \mathbf{n}_{\mathrm{BE}} &= -\mathbf{k}, & \mathbf{n}_{\mathrm{OE}} &= \mathbf{i} \\ \mathbf{n}_{\mathrm{OB}} &= 0.6\mathbf{i} + 0.8\mathbf{k}, & \mathbf{n}_{\mathrm{OD}} &= 0.6\mathbf{i} + 0.8\mathbf{j} \end{aligned}$$

Next, we choose to take the moment about line $\overline{OB}$ to determine T_{AC}.

$$[\textstyle\sum M_{\mathrm{OB}} = 0] \qquad [(-0.75\mathbf{i}) \times (-400g\mathbf{j}) + (2\mathbf{k}) \times T_{\mathrm{AC}}\mathbf{n}_{\mathrm{AC}}] \cdot \mathbf{n}_{\mathrm{OB}} = 0$$

$$T_{\mathrm{AC}} = 4808.8 \ \ \mathrm{N}$$

Then, take the moment about line $\overline{OD}$ to determine T_{BE}.

$$[\textstyle\sum M_{\mathrm{OD}} = 0]$$

$$[(2\mathbf{k}) \times T_{\mathrm{AC}}\mathbf{n}_{\mathrm{AC}} + (0.75\mathbf{i} + 2\mathbf{k}) \times (-400g\mathbf{j}) + (1.5\mathbf{i}) \times T_{\mathrm{BE}}\mathbf{n}_{\mathrm{BE}}] \cdot \mathbf{n}_{\mathrm{OD}} = 0$$

$$T_{\mathrm{BE}} = 654 \ \ \mathrm{N}$$

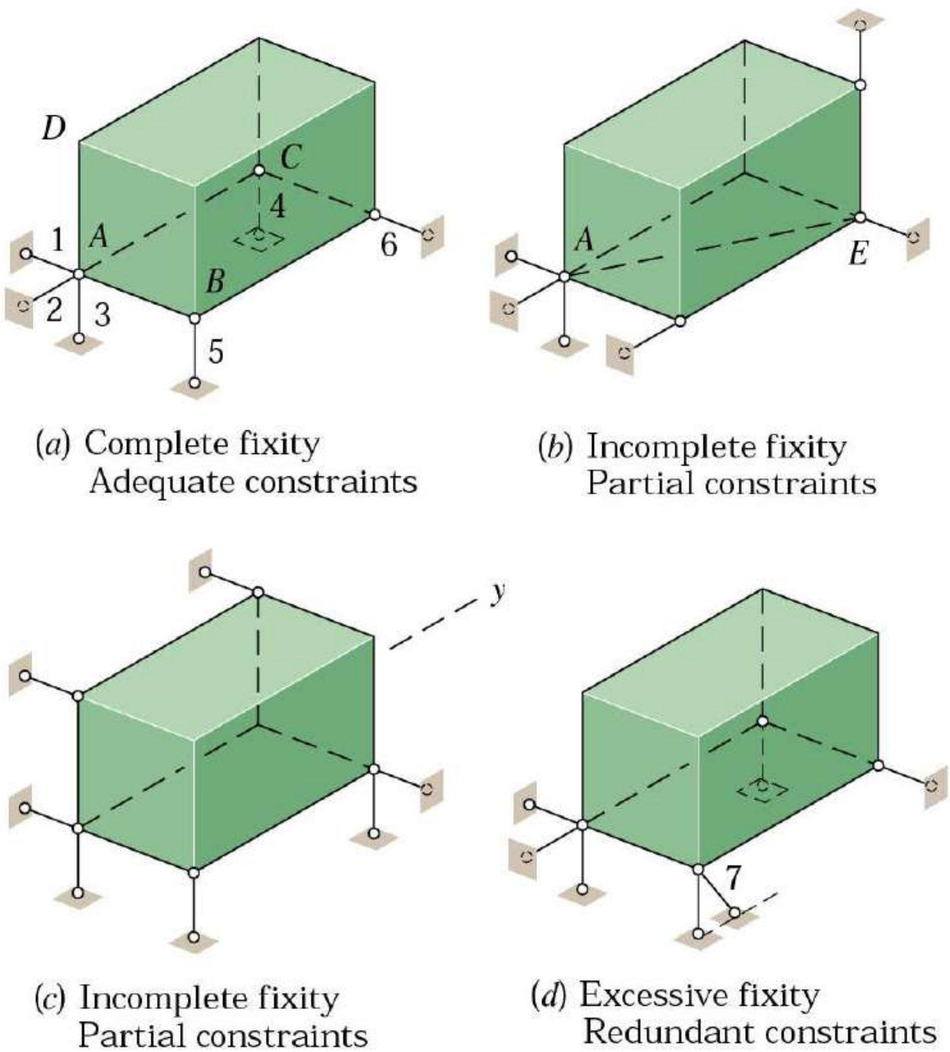


Figure 3.40: Three Dimensional Constraints and Statical Determinacy ([1], pp. 149)

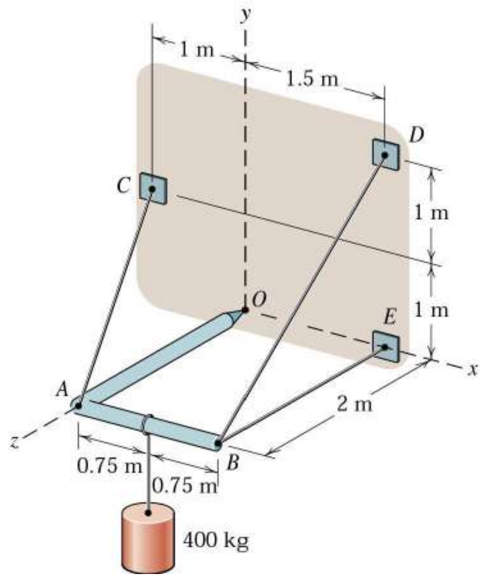


Figure 3.41: Example 3.11 ([2], pp. 151)

The last cable tension T_{BD} is found from taking the moment about line $\overline{OE}$.

$[\sum M_{\mathrm{OE}} = 0]$

$$[(2\mathbf{j}) \times T_{\mathrm{BD}}\mathbf{n}_{\mathrm{BD}} + (0.75\mathbf{i} + 2\mathbf{k}) \times (-400g\mathbf{j}) + (2\mathbf{k}) \times T_{\mathrm{AC}}\mathbf{n}_{\mathrm{AC}}] \cdot \mathbf{n}_{\mathrm{OE}} = 0$$

$$T_{\mathrm{BD}} = 2775.1 \text{ N}$$

Determining the reactions at O is rather straightforward. The sum of the forces acting on the system in any direction must be zero.

$[\sum \mathbf{F} = \mathbf{0}]$ $\quad O_x = 1962 \text{ N}, \quad O_y = 0 \text{ N}, \quad O_z = 6540 \text{ N}$

Example 3.12 ([1], Prob. 3/67) The 600 kg industrial door is a uniform rectangular panel which rolls along the fixed rail D on its hanger-mounted wheels A and B. The door is maintained in a vertical plane by the floor-mounted guide roller C, which bears against the bottom edge. For the position shown compute the horizontal side thrust on each of the wheels A and B, which must be accounted for in the design of the brackets.

Solution: The FBD of the door including the hanger A and B is shown in fig. 3.44. Due to the offset of the door's weight and the normal forces at the

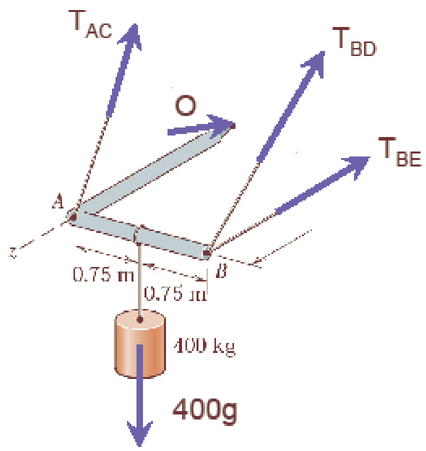


Figure 3.42: Solution to example 3.11

wheels, the normal force at the guide rollor C points in the direction shown to counter for the induced moment. Hence the side thrusts on the wheels must point in the opposite direction to balance the force.

$$[\sum M_{\mathrm{AB}} = 0] \qquad 600g \times 0.15 - N_{\mathrm{C}} \times 3 = 0, \quad N_{\mathrm{C}} = 294.3 \ \ \mathrm{N}$$

$$[\sum M_{\mathrm{A}_z} = 0] \qquad N_{\mathrm{C}} \times 0.6 - B_x \times 3 = 0, \quad B_x = 58.86 \ \ \mathrm{N}$$

$$[\sum F_x = 0] \qquad A_x + B_x - N_{\mathrm{C}} = 0, \quad A_x = 235.44 \ \ \mathrm{N}$$

Example 3.13 ([1], Prob. 3/73) The smooth homogeneous sphere rests in the 120° groove and bears against the end plate which is normal to the direction of the groove. Determine the angle θ, measured from the horizontal, for which the reaction on each side of the groove equals the force supported by the end plate.

Solution: First the FBD will be drawn according to the given condition that the reaction on each side of the groove must equal the force supported by the end plate. However, it is rather difficult to draw and visualize the FBD of the sphere in three dimensions. Therefore the FBD will be drawn in two orthogonal views; along and perpendicular to the V-groove. Forces acting on the sphere will then be the projected components onto that view. Also the coordinate frame x-y-z is set at the center of the sphere for convenience. See fig. 3.46.

Now the equilibrium conditions are ready to be applied under the condition that $N_1 = N_2 = N_{\mathrm{r}} = N$.

$$[\sum F_y = 0] \qquad N_1 = N_2 = N$$

$$[\sum F_z = 0] \qquad mg\cos\theta = 2N\cos 30$$

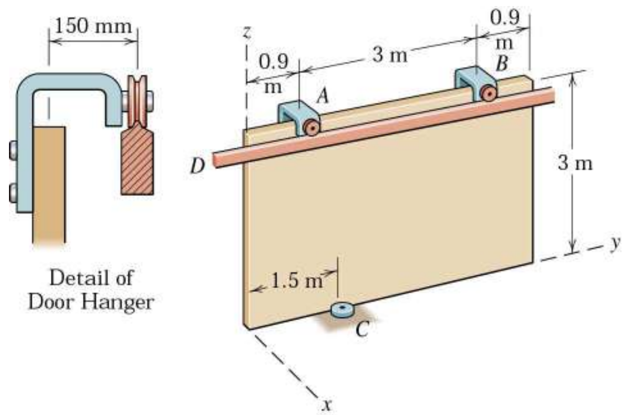


Figure 3.43: Example 3.12 ([1], pp. 154)

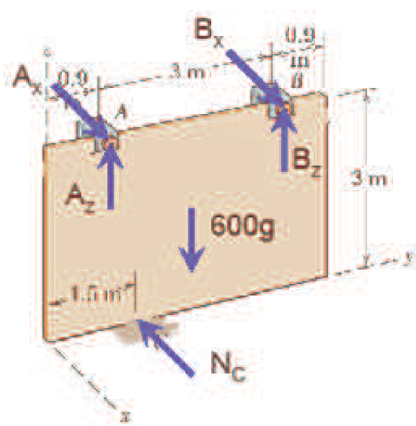


Figure 3.44: Solution to example 3.12

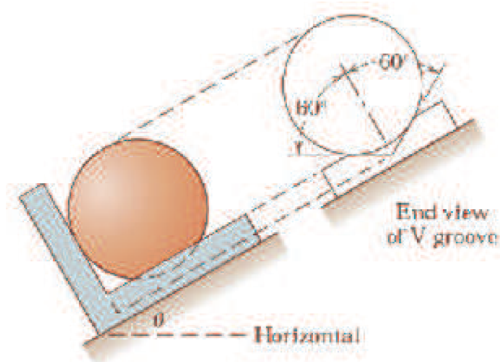


Figure 3.45: Example 3.13 ([1], pp. 156)

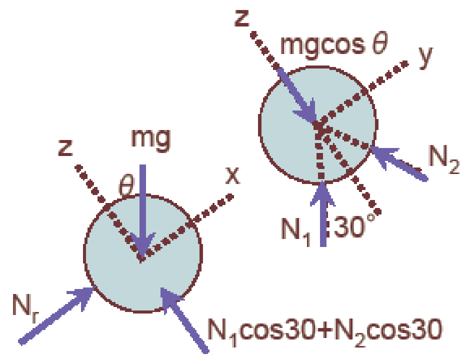


Figure 3.46: Solution to example 3.13

$[\sum F_x = 0]$ $\qquad N_{\rm r} = mg\sin\theta$

Under the condition $N_{\rm r} = N$,

$$\tan\theta = \frac{1}{2}\cos 30 \quad \Rightarrow \quad \theta = 30°, N = mg/2$$

Example 3.14 ([1], Prob. 3/75) The mass center of the 30 kg door is in the center of the panel. If the weight of the door is supported entirely by the lower hinge A, calculate the magnitude of the total force supported by the hinge at B.

Solution: From the problem statement, it is implied that there is no vertical support force at the upper hinge B. This assumption must have been enforced otherwise it cannot be solved by just the equilibrium conditions. The FBD is straightforward with the five unknown supports revealed and the appropriate coordinate frame set. Now it is ready to apply the equilibrium conditions:

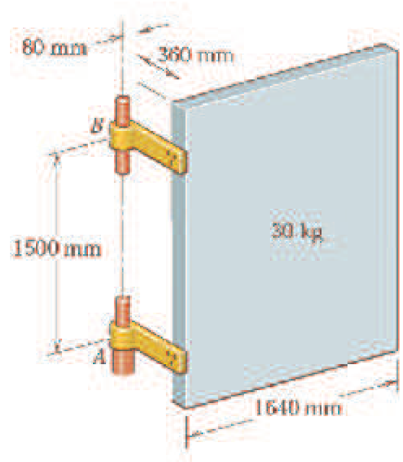


Figure 3.47: Example 3.14 ([1], pp. 156)

$[\sum F_x = 0]\,[\sum M_{A_y} = 0] \qquad 30g \times 0.36 - B_x \times 1.5 = 0, \quad B_x = A_x = 70.6 \;\; \text{N}$

$[\sum F_y = 0]\,[\sum M_{A_x} = 0] \qquad B_y \times 1.5 - 30g \times 0.9 = 0, \quad B_y = A_y = 176.6 \;\; \text{N}$

$$B = \sqrt{B_x^2 + B_y^2} = 190.2 \;\; \text{N}$$

Example 3.15 ([1], Prob. 3/82) One of the three landing pads for the Mars Viking lander is shown in the figure with its approximate dimensions. The mass of the lander is 600 kg. Compute the force in each leg when the lander is resting on a horizontal surface on Mars. Assume equal support by the pads and consult table D/2 in Appendix D as needed.

Solution: From the problem statement, it can be deduced that each landing pad supports 200 kg force. However the gravitational constant on Mars is $g = 3.73\text{m/s}^2$. The leg with the landing pad is sectioned out and its FBD is drawn. Forces in each strut are then revealed. Note that the forces are all concurrent at the center of ball and socket. Hence the available equilibrium conditions are reduced to three only.

First the relating unit vectors are determined:

$$\mathbf{n}_{\text{DC}} = 0.35\mathbf{i} - 0.936\mathbf{k}, \;\; \mathbf{n}_{\text{CA}} = -0.7664\mathbf{i} + 0.418\mathbf{j} + 0.4877\mathbf{k}$$

Next, the moment equilibrium is taken about $\overline{BA}$ to determine $\mathbf{F}_{\text{DC}}$ directly.

$[\sum M_{\text{BA}} = 0] \qquad [(0.85\mathbf{k} + 0.1\mathbf{i}) \times F_{\text{DC}}\mathbf{n}_{\text{DC}} - 200g \times 0.55\mathbf{j}] \cdot \mathbf{j} = 0$

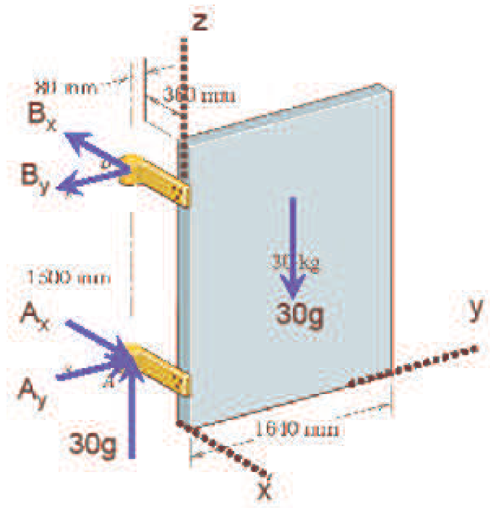


Figure 3.48: Solution to example 3.14

$$F_{\mathrm{DC}} = 1049.1 \;\; \mathrm{N}$$

Finally, apply the force equilibrium conditions, or using the symmetry of the leg about x-z plane,

$$[\sum F_x = 0]\,[\sum F_y = 0] \qquad F_{\mathrm{DC}}\mathbf{n}_{\mathrm{DC}}\cdot\mathbf{i} - 2T_{\mathrm{CA}}\times 0.7664 = 0 \;\Rightarrow\; T_{\mathrm{CA}} = T_{\mathrm{CB}} = 239.5 \;\; \mathrm{N}$$

Example 3.16 ([2], Prob. 3/88) The uniform 15 kg plate is welded to the vertical shaft, which is supported by bearings A and B. Calculate the magnitude of the force supported by bearing B during application of the 120 Nm couple to the shaft. The cable from C to D prevents the plate and shaft from turning, and the weight of the assembly is carried entirely by bearing A.

Solution: The plate and the attached vertical shaft are selected as the system and its FBD is drawn. By the statement that the weight of the assembly is carried entirely by bearing A, bearing B will experience only the radial forces. Since the only vertical force is the assembly's weight, the vertical component reaction at the bearing A must be equal to the weight. The convenient coordinate frame is defined and now it is ready to set up the equilibrium conditions.

$$\mathbf{n}_{\mathrm{DC}} = -0.95\mathbf{i} - 0.316\mathbf{j}$$

First, take moment about the z-axis so all unknowns except the tension T are eliminated.

$$[\sum M_{\mathrm{O}_z} = 0] \qquad 120 + 0.6\mathbf{i}\times T\mathbf{n}_{\mathrm{DC}}\cdot\mathbf{k} = 0, \quad T = 632.9 \;\; \mathrm{N}$$

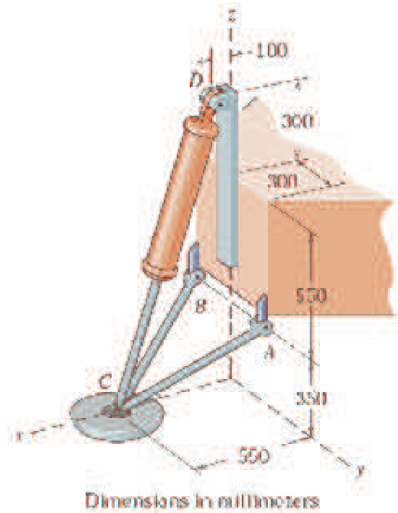


Figure 3.49: Example 3.15 ([1], pp. 158)

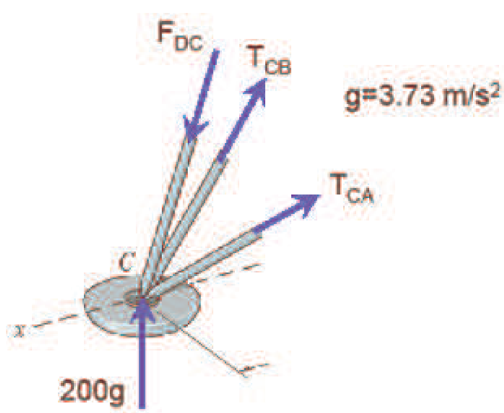


Figure 3.50: Solution to example 3.15

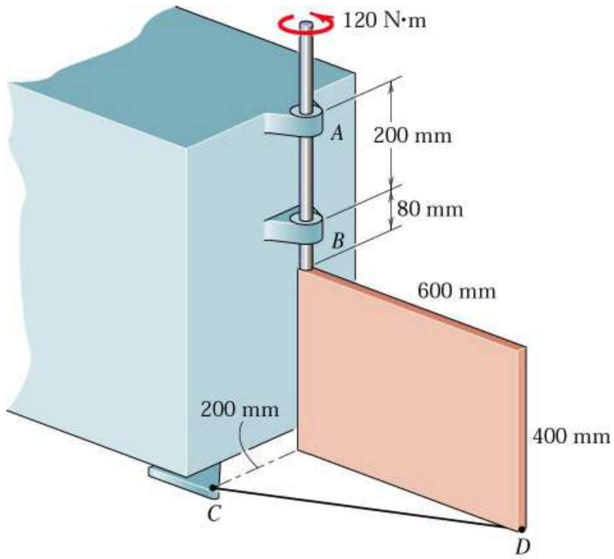


Figure 3.51: Example 3.16 ([2], pp. 157)

Then, take moment about the line in y-direction passing A so the reaction forces at A and B_y are eliminated.

$$\left[\sum M_{\mathrm{A_y}} = 0\right] \qquad -B_x \times 0.2 + 15g \times 0.3 + T_x \times 0.68 = 0, \quad B_x = 2265 \ \ \mathrm{N}$$

Finally, take moment about the line in x-direction passing A so B_y can be determined.

$$[\sum M_{\mathrm{A_x}} = 0] \qquad B_y \times 0.2 - T_y \times 0.68 = 0, \quad B_y = 680 \ \ \mathrm{N}$$

$$B = \sqrt{B_x^2 + B_y^2} = 2635 \ \ \mathrm{N}$$

Example 3.17 ([2], Prob. 3/95) The uniform 900 × 1200 mm trap door has a mass of 200 kg and is propped open by the light strut AB at the angle $\theta = \text{atan}\left(\frac{4}{3}\right)$. Calculate the compression F_B in the strut and the force supported by the hinge D normal to the hinge axis. Assume that the hinges act at the extreme ends of the lower edge.

Solution: The trap door is isolated and its FBD is drawn. There are the reactions at C and D, the strut force, and its weight. Reactions at D are decomposed along the coordinate frame axes. It is unnecessary to do so for the reactions at C, however.

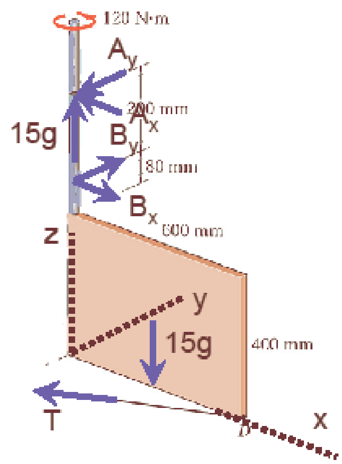


Figure 3.52: Solution to example 3.16

Since the strut lies along the obligue line, it is helpful to determine the directional unit vector.

$$\mathbf{n}_{\text{AB}} = -0.2857\mathbf{i} - 0.4286\mathbf{j} + 0.857\mathbf{k}$$

The moment equilibrium conditions are applied at C so no reaction force at C appears. Along the x-axis,

$$[\sum M_{\text{C}_\text{x}} = 0] \qquad (0.9\mathbf{j} \times T_{\text{AB}}\mathbf{n}_{\text{AB}}) \cdot \mathbf{i} - 200g \times 0.45 \cos 53.13 = 0, \quad T_{\text{AB}} = 688 \text{ N}$$

Take the moment along the y-axis so only unknown D_z remains:

$$\left[\sum M_{\text{C}_\text{y}} = 0\right] \qquad -200g \times 0.6 + D_z \times 1.2 = 0, \quad D_z = 981 \text{ N}$$

About the z-axis, all unknowns except D_y are eliminated:

$$[\sum M_{\text{C}_\text{z}} = 0] \qquad -D_y \times 1.2 + (-T_{\text{AB}}\mathbf{n}_{\text{AB}} \cdot \mathbf{i}) \times 0.9 = 0, \quad D_y = 147.4 \text{ N}$$

Therefore, the supporting force at hinge D normal to its axis is

$$D_{\text{n}} = \sqrt{D_y^2 + D_z^2} = 992 \text{ N}$$

Example 3.18 ([1], Prob. 3/95) The uniform rectangular panel $ABCD$ has a mass of 40 kg and is hinged at its corner A and B to the fixed vertical surface. A wire from E to D keeps edges BC and AD horizontal. Hinge A can support thrust along the hinge axis AB, whereas hinge B supports force normal to the hinge axis only. Find the tension T in the wire and the magnitude B of the force

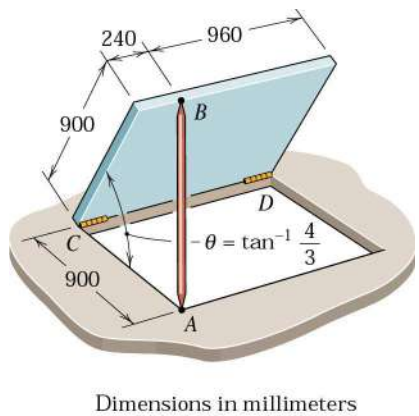


Figure 3.53: Example 3.17 ([2], pp. 160)

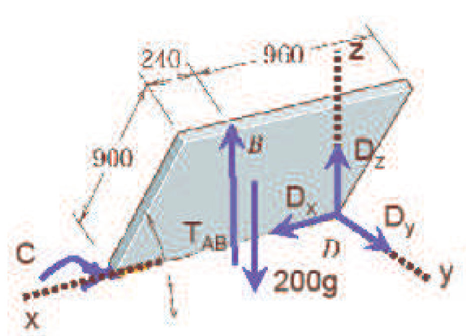


Figure 3.54: Solution to example 3.17

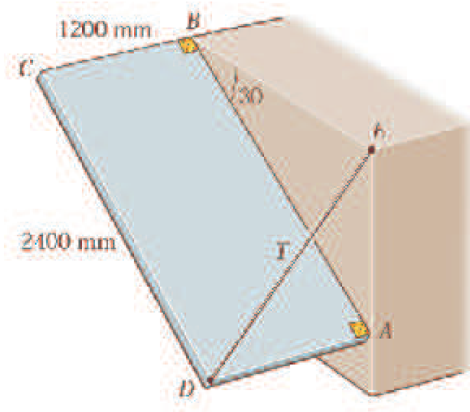


Figure 3.55: Example 3.18 ([1], pp. 162)

supported by hinge B.

Solution: The FBD of the panel $ABCD$ is shown in fig. 3.56. Note that there is no reaction component along the hinge axis at B. Assigning the coordinate frame x-y-z as shown,

$$\mathbf{n}_{\mathrm{DE}} = 0.35\mathbf{i} - 0.707\mathbf{j} + 0.61\mathbf{k}$$

Take moment about the x-axis so only the tension force remains:

$$[\sum M_{\mathrm{A_x}} = 0] \qquad [0.6\mathbf{j} \times 40g\,(-\cos 30\mathbf{k} - \sin 30\mathbf{i})] \cdot \mathbf{i} + (1.2\mathbf{j} \times T_{\mathrm{DE}}\mathbf{n}_{\mathrm{DE}}) \cdot \mathbf{i} = 0$$

$$T_{\mathrm{DE}} = 278.55 \text{ N}$$

Next, take moment about point A along the y-axis. Only B_z then appears in the equation.

$$\left[\sum M_{\mathrm{A_y}} = 0\right] \qquad [1.2\mathbf{i} \times 40g\,(-\cos 30\mathbf{k} - \sin 30\mathbf{i})] \cdot \mathbf{j} - 2.4B_z = 0$$

$$B_z = 169.9 \text{ N}$$

Finally, take moment about the vertical line AE. Reaction forces at A, the tension force, the panel's weight, and B_z do not contribute any moment. Therefore

$$[\sum M_{\mathrm{AE}} = 0] \qquad B_y = 0 \text{ N} \;\Rightarrow\; B_{\mathrm{n}} = 169.9 \text{ N}$$

Example 3.19 ([2], Prob. 3/98) Under the action of the 40 Nm torque (couple) applied to the vertical shaft, the restraining cable AC limits the rotation of

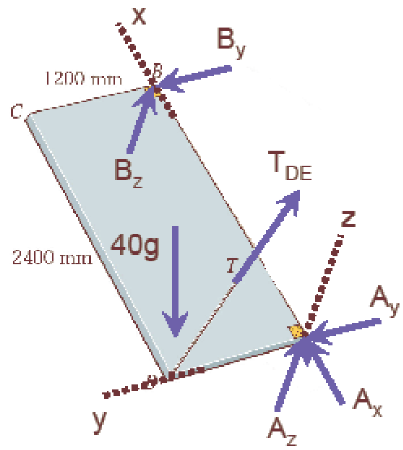


Figure 3.56: Solution to example 3.18

the arm OA and attached shaft to an angle of 60° measured from the y-axis. The collar D fastened to the shaft prevents downward motion of the shaft in its bearing. Calculate the bending moment M, the compression P, and the shear force V in the shaft at section B. (note: Bending moment, expressed as a vector, is normal to the shaft axis, and shear force is also normal to the shaft axis.)

Solution: The shaft is sectioned at B (fig. 3.58) so that the reaction forces and moments are exposed. Reactions are the compressive force, the shear force, and the bending moment. However the shaft is free to rotate about its axis. Hence no supporting moment in this direction.

$$\mathbf{n}_{\text{AC}} = 0.53\mathbf{i} + 0.38\mathbf{j} - 0.758\mathbf{k}$$

Take moment about the z-axis, so the tension force can be determined.

$$[\sum M_z = 0] \qquad 40 + (0.18\mathbf{j} \times T_{\text{AC}}\mathbf{n}_{\text{AC}}) \cdot \mathbf{k} = 0 \quad \Rightarrow \quad T_{\text{AC}} = 419.3 \text{ N}$$

The compressive force can then be determined by the force equilibrium condition in the vertical axis.

$$[\sum F_z = 0] \qquad P + T_{\text{AC}}\mathbf{n}_{\text{AC}} \cdot \mathbf{k} = 0 \quad \Rightarrow \quad P = 317.8 \text{ N}$$

The shear force is the remaining of the tension force after subtracting the compressive force.

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{T_{\text{AC}}^2 - P^2} = 273.5 \text{ N}$$

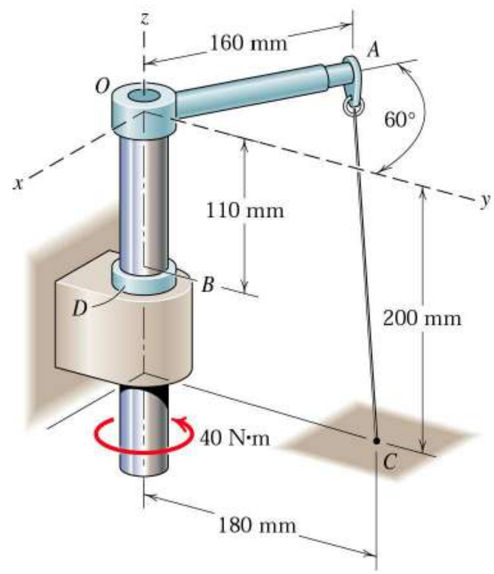


Figure 3.57: Example 3.19 ([2], pp. 161)

$[\sum \mathbf{M}_B = \mathbf{0}] \qquad M_{\mathrm{B_x}}\mathbf{i} + M_{\mathrm{B_y}}\mathbf{j} + 40 + (-0.09\mathbf{k} + 0.18\mathbf{j}) \times T_{\mathrm{AC}}\mathbf{n}_{\mathrm{AC}} = \mathbf{0}$

$$M_{\mathrm{B_x}} = 42.87 \ \ \mathrm{Nm}, M_{\mathrm{B_y}} = 20.0 \ \ \mathrm{Nm} \qquad M_{\mathrm{b}} = \sqrt{M_{\mathrm{B_x}}^2 + M_{\mathrm{B_y}}^2} = 47.3 \ \ \mathrm{Nm}$$

Example 3.20 ([1]) Determine the turning force at the knob DE and the normal forces at each leg.

Solution: The reel is supported by the structure that is connected to the reel axis. The turning force can be determined from the FBD of the reel only (left of fig. 3.60). The reaction forces and the bending moments are exposed. Since these forces are not of interest, take moment about the y-axis.

$[\sum M_y = 0] \qquad 100 \times 0.15 - P \times 0.3 = 0, \quad P = 50 \ \ \mathrm{N}$

To determine the normal forces at each leg, consider the FBD of the whole assembly shown on the right of fig. 3.60. Note of the friction forces which are not negligible. Taking moment at B along the x-axis,

$[\sum M_{\mathrm{B_x}} = 0] \qquad 200 \times 0.2 - 100 \sin 15 \times 0.35 - P \sin 30 \times 0.075 - N_C \times 0.5 = 0, \quad N_C = 58.13 \ \ \mathrm{N}$

Taking moment at B along the y-axis,

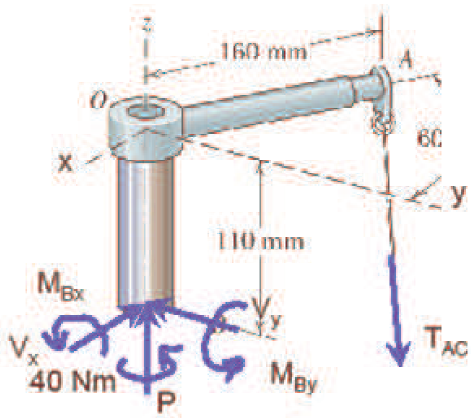


Figure 3.58: Solution to example 3.19

$[\sum M_{B_y} = 0]$

$$-N_A \times 0.525 - N_C \times 0.2625 + 200 \times 0.2625 + 100 \cos 15 \times 0.52$$

$$-100 \sin 15 \times 0.2237 - P \cos 30 \times 0.635 + P \sin 30 \times 0.1125 = 0$$

$$N_A = 108.56 \text{ N}$$

Lastly, use the force equilibrium condition in the vertical direction to determine the remaining N_B.

$[\sum F_z = 0]$ $\quad N_A + N_B + 100 \sin 15 + N_C - P \sin 30 - 200 = 0, \quad N_B = 32.44 \text{ N}$

Example 3.21 ([1], Prob. 3/114) The drum and shaft are welded together and have a mass of 50 kg with mass center at G. The shaft is subjected to a torque (couple) of 120 Nm, and the drum is prevented from rotating by the cord wrapped securely around it and attached to point C. Calculate the magnitudes of the forces supported by bearings A and B.

Solution: Select the drum and shaft unit and draw the FBD. Each bearing supports the shaft by the radial force which is projected into the x- and z-direction as shown in fig. 3.62.

First, take moment about B along the y-axis.

$[\sum M_{B_y} = 0]$ $\quad T \times 0.15 - 120 = 0, \quad T = 800 \text{ N}$

Then, take moment about B along the z- and x-axes to obtain the bearing forces at A.

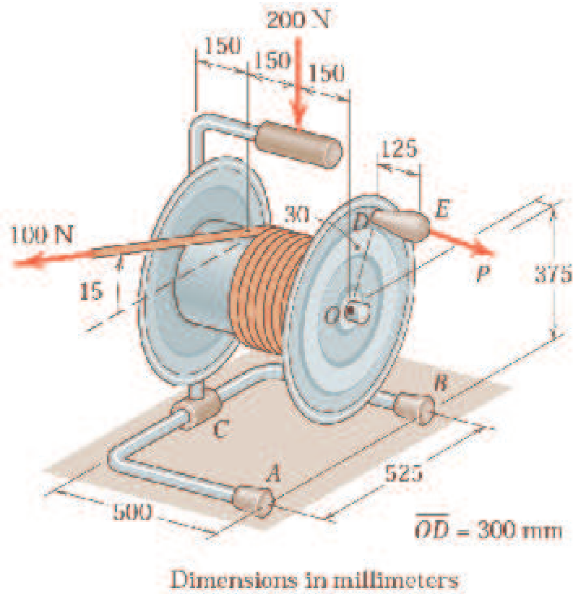


Figure 3.59: Example 3.20 ([1])

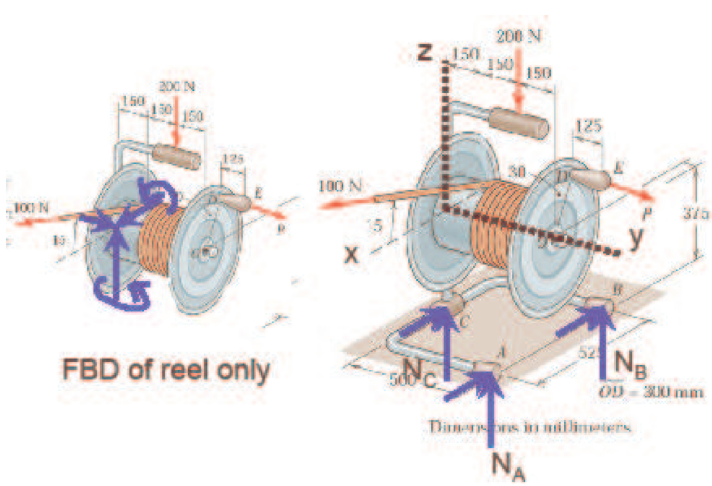


Figure 3.60: Solution to example 3.20

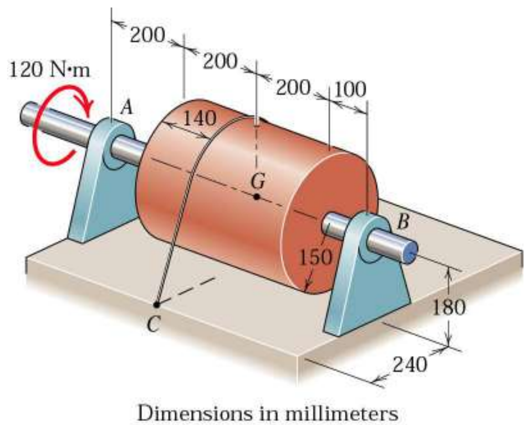


Figure 3.61: Example 3.21 ([1], pp. 168)

$[\sum M_{B_z} = 0]$ $\quad T\cos 66.87 \times 0.36 - A_x \times 0.7 = 0, \quad A_x = 161.6 \text{ N}$

$[\sum M_{B_x} = 0]$ $\quad 50g \times 0.3 + T\sin 66.87 \times 0.36 - A_z \times 0.7 = 0, \quad A_z = 588.6 \text{ N}$

Finally, use the force equilibrium conditions to determine the bearing forces at B.

$[\sum F_x = 0]$ $\quad A_x + B_x - T\cos 66.87 = 0, \quad B_x = 152.6 \text{ N}$

$[\sum F_z = 0]$ $\quad A_z + B_z - 50g - T\sin 66.87 = 0, \quad B_z = 637.6 \text{ N}$

Therefore,

$$A = \sqrt{A_x^2 + A_z^2} = 610.4 \text{ N}, \quad B = \sqrt{B_x^2 + B_z^2} = 655.6 \text{ N}$$

Example 3.22 ([1], Prob. 3/115) Determine the reaction force and moment at the double U-joint support O.

Solution: By the double U-joint mechanism, O can provide support forces in any direction. However, it can only support the moment in the direction normal to the back plate. The FBD of the boom system is shown in fig. 3.64. The unit vector along the cable direction are

$$\mathbf{n}_{BC} = 0.13\mathbf{i} - 0.91\mathbf{j} + 0.39\mathbf{k}, \quad \mathbf{n}_{AD} = -0.48\mathbf{i} - 0.84\mathbf{j} + 0.241\mathbf{k}$$

Only O_z remains when formulating the moment equation about AB:

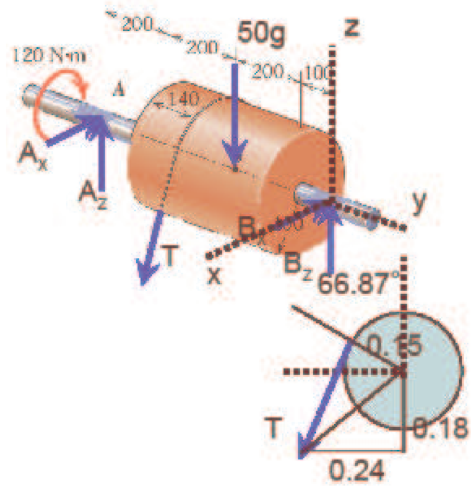


Figure 3.62: Solution to example 3.21

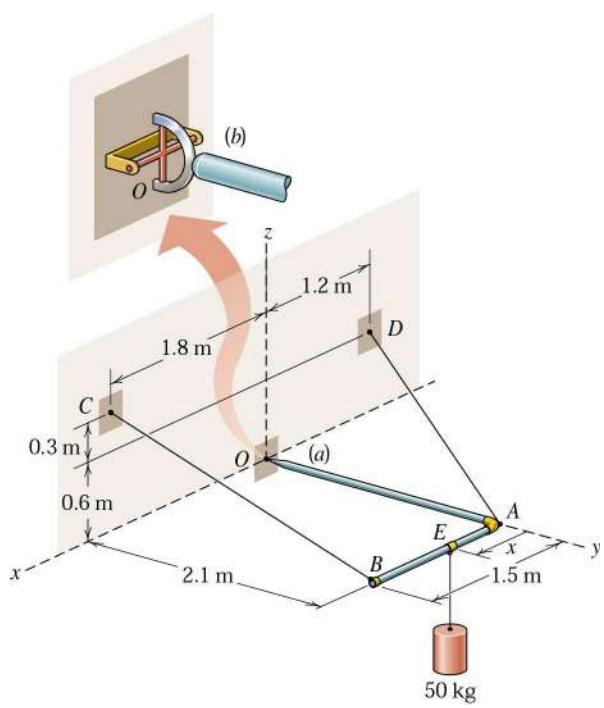


Figure 3.63: Example 3.22 ([1], pp. 168)

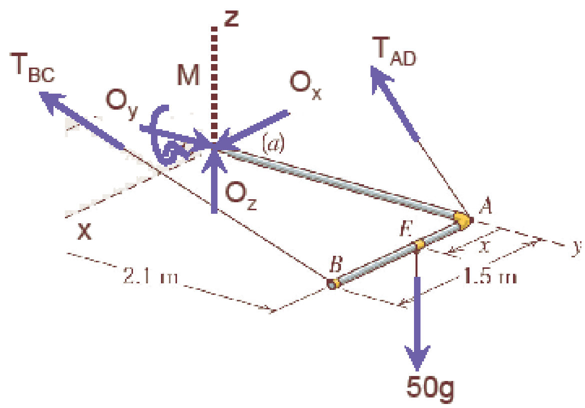


Figure 3.64: Solution to example 3.22

$[\sum M_{\text{AB}} = 0]$ $$O_z = 0 \text{ N}$$

To determine the cable forces, take the moment about O normal to the y-axis because the unknown moment M always lies along the y-axis.

$[\sum M_z = 0]$ $$(1.8\mathbf{i} \times T_{\text{BC}}\mathbf{n}_{\text{BC}}) \cdot \mathbf{k} + (2.1\mathbf{j} \times T_{\text{AD}}\mathbf{n}_{\text{AD}}) \cdot \mathbf{k} = 0$$

$[\sum M_x = 0]$ $$(2.1\mathbf{j} \times T_{\text{BC}}\mathbf{n}_{\text{BC}}) \cdot \mathbf{i} + (2.1\mathbf{j} \times T_{\text{AD}}\mathbf{n}_{\text{AD}}) \cdot \mathbf{i} - 50g \times 2.1 = 0$$

$$T_{\text{BC}} = 625 \text{ N}, \quad T_{\text{AD}} = 1024 \text{ N}$$

Now the moment equilibrium about y-axis will give the supporting moment at the double U-joint.

$[\sum M_y = 0]$ $$M + 50g \times x + (1.5\mathbf{i} \times T_{\text{BC}}\mathbf{n}_{\text{BC}}) \cdot \mathbf{j} = 0$$

$$M = 365.66 - 490.5x$$

Finally, the force equilibrium along the x- and y- axis will help determine the remaining supporting forces.

$[\sum F_y = 0]$ $$O_y + T_{\text{BC}}\mathbf{n}_{\text{BC}} \cdot \mathbf{j} + T_{\text{AD}}\mathbf{n}_{\text{AD}} \cdot \mathbf{j} = 0, \quad O_y = 1429 \text{ N}$$

$[\sum F_x = 0]$ $$O_x + T_{\text{BC}}\mathbf{n}_{\text{BC}} \cdot \mathbf{i} + T_{\text{AD}}\mathbf{n}_{\text{AD}} \cdot \mathbf{i} = 0, \quad O_x = 410 \text{ N}$$

$$O = \sqrt{O_x^2 + O_y^2 + O_z^2} = 1487 \text{ N}$$

Chapter 4

Friction

4.1 Introduction

When two objects are in contact, the forces of action and reaction between contacting surfaces are developed. To the mutual contacting surface, these forces have their components both in the tangential and normal directions. Force component in the tangential direction is known as the *friction* force. **Whenever a tendency exists for one contacting surface to slide along another surface, the developed friction force is always in the direction opposing this tendency**.

In some systems, friction is undesirable because it just plainly changes the system characteristics from the required behavior. In particular, where the sliding motion between parts occurs, the developed friction force results in a loss of energy. However, in many cases, friction instead functions the systems. For example, many mechanisms employ friction as an extra force to retain their equilibrium states.

4.2 Types of Friction

4.3 Dry Friction

4.3.1 Mechanism of Friction

To understand the friction phenomena, which is important in further problem analysis, let us consider fig. 4.1. A block of mass m slides on a planar surface with the applied pulling force P, as shown in fig. 4.1(a). To begin the analysis, the free body diagram is drawn, depicted in fig. 4.1(b). The reaction force R the ground exerted on the block is decomposed into the normal and tangential components, according to the virtual tangent plane at the contact. The tangential component or the friction force, denoted F, is related to the normal force N by the trigonometrical relationship

$$F = N \tan \alpha \tag{4.1}$$

α, called the *friction angle*, is the angle the resulting reaction force deviated from the normal direction. This angle can be determined approximately by the experiment. The value depends largely on the materials of two contacting surface. Fig. 4.1(c) explains why the parameter is at best obtained as an estimated value. Local geometry of the contact point determines each infinitesimal reaction force which adds up vectorially to give the gross reaction force R. A multitude of parameters dictate the value of the macro friction angle α, which therefore is the source of uncertainties.

Relationship of the friction force F and the applied force P is plotted in fig. **??**(d). There are three regimes of the friction force development. In the

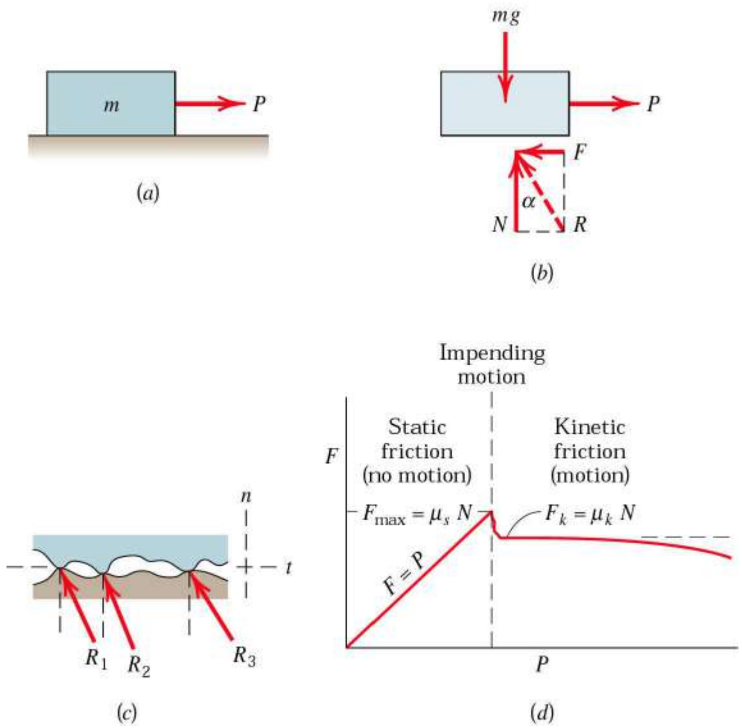


Figure 4.1: Mechanism of the friction ([1], pp. 341)

beginning of the application, the friction force equally catch up the applied force as shown by the 45° straight line dictating $F = P$. During this phase, the block does not move at all. This can be understood by determining the resulting force acting along the tangential direction. The force is zero which indicates the equilibrium status and hence *no motion* of the block.

The first regime lasts until the applied force P reaches

$$F_{\max} = \mu_s N \tag{4.2}$$

for which the graph enters the second regime. μ_s is the static coefficient of friction. By comparing eq. 4.1 and 4.2,

$$\mu_s = \tan \alpha_s \tag{4.3}$$

Hence the coefficient of friction depends on the material of the mating surfaces. At this point, the block is about to move, called the *impending motion*, because the current applied force is the largest force resistable by the physically largest generated friction force $F_{\max}$, governed by eq. 4.2. From the equation, the maximum friction force depends on the normal force and the static coefficient of friction. The implication of the impending motion is the validity of the equilibrium condition and the friction force reaches the maximum value.

If the applied force is further increased, the block is no longer in the equilibrium. That is the frictional force is now not enough to hold the block at rest. From the unbalanced resulting force, the block will then start *moving* in the direction of the applied force. Moreover, the magnitude of the friction force itself is known to be dropped off from the maximum value $F_{\max}$. The new value of the friction in the third regime is

$$F_k = \mu_k N \tag{4.4}$$

where μ_k is the kinetic coefficient of friction determined experimentally. Its value is usually less than μ_s and also depends on the material types of the contact surfaces. Practically, the friction force decreases as the relative velocity of the mating surfaces increases.

In summary, there are three regions of the friction force development in transitioning of the object from rest to motion. These are

1. *No Motion* is the region up to the point of slippage or impending motion. Friction force is determined by the equations of equilibrium because *the system is in equilibrium*. When the motion is not impending, $F < F_{\max}$.

2. *Impending Motion* is the moment where the body is on the verge of slipping. *Static friction force reaches the maximum value*. For a given pair of mating surfaces, $F = F_{\max} = \mu_s N$.

3. *Motion* The body starts moving in the direction of the applied force. Here, *friction force drops to a lower value called kinetic friction* $F = \mu_k N$. It will drop further with higher velocity.

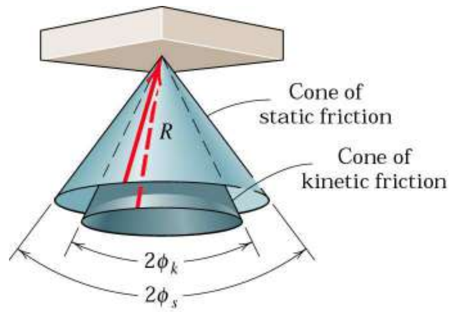


Figure 4.2: Friction cone ([1], pp. 344)

4.3.2 Friction Cone

The friction coefficient reflects the roughness of a pair of mating surfaces. The smaller the value, the smoother the surfaces and the easier to move relatively to each other. Furthermore, this coefficient has another interpretation according to eq. 4.3. The angle α_s restricts the deviation angle that the actual reaction force R made with the normal direction. See fig. 4.2. The direction of the reaction force R is specified by $\tan\alpha = F/N$. It will lie inside the so called *cone of friction*, because any valid friction force is limited by $F_{\max} = \tan\mu_s N$. When the friction force reaches the maximum value, i.e. $\tan\alpha = \tan\alpha_s = \mu_s$, the reaction force R lies on the surface of the *static* cone. In other words, the direction of the reaction force is known in case of the impending motion. This is helpful in solving the problem as will be illustrated in some examples. Corollarily, there is the smaller *kinetic* cone of friction to guide the reaction force direction when the objects' surfaces are slipping relative to each other. The relationship becomes $\tan\alpha = \tan\alpha_k = \mu_k$. Note the friction force is independent of the apparent or projected area of contact.

4.3.3 Solving the Dry Friction Problems

Here the general approach in solving the dry friction problems will be given. It involves the identification of the status of the motion that influences the determination of the friction force.

Condition of impending motion is known to exist The body is in equilibrium and on the verge of slipping. The friction force is therefore the maximum static friction $F = \mu_s N$.

Relative motion is known to exist In this case the friction force is the kinetic friction force $F = \mu_k N$.

Unknown status of the problem This is the difficult case. The problem is tackled by *first assuming the system is in static equilibrium and using the equilibrium condition to determine the required friction force F*. The result is then investigated to conclude the validity of the equilibrium assumption:

1. $F < \mu_s N$ indicates that the friction force for the assumed equilibrium condition can be provided and so the body is in static equilibrium as firstly assumed.

2. $F = \mu_s N$ implies the maximum friction force is required. The assumed static equilibrium condition still holds and so the motion impends.

3. $F > \mu_s N$ This is impossible because the surface cannot support more friction than $\mu_s N$. Consequently, the equilibrium assumption is invalid and motion occurs instead. This makes the friction force be directed by the kinetic friction force; $F = \mu_k N$. And the equilibrium conditions are no longer held. The motion is accelerated.

Using the above guidelines, the following representative problems are now presented and solved.

Example 4.1 ([1], SP. 6/1) Determine the maximum angle θ which the adjustable incline may have with the horizontal before the block of mass m begins to slip. The coefficient of static friction between the block and the inclined surface is μ_s.

Solution: From the given statement, the block is on the verge of slipping. Therefore the friction force is the maximum value $F = \mu_s N$ acting upward the incline. According to the free body diagram in fig. 4.4, the following equilibrium equations can be formulated.

$$[\sum F_y = 0] \qquad N - mg\cos\theta = 0$$

$$[\sum F_x = 0] \qquad \mu_s N - mg\sin\theta = 0$$

From the above equations, it can be concluded that

$$\mu_s = \tan\theta \quad \text{or} \quad \theta = \tan^{-1}\mu_s$$

Another approach in solving this problem can be done by employing the concept of the friction cone. From the right free body diagram shown in fig. 4.4, the equilibrium simply requires the balance between the reaction force R and the weight W. Recognizing that the reaction force will make the angle of $\alpha_s =$

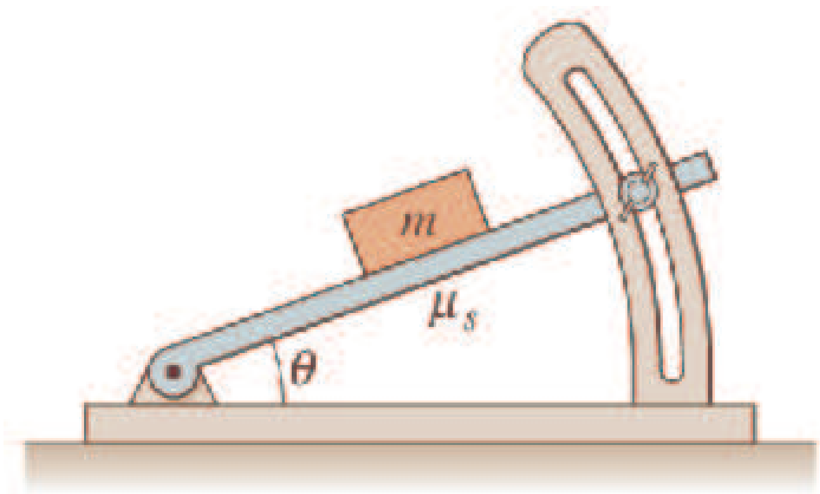


Figure 4.3: Example 4.1 ([1], pp. 346)

$\tan^{-1}\mu_s$ to the normal line when the motion impends, and noting that from the geometry $\alpha = \theta$, it then can be concluded that

$$\theta = \tan^{-1}\mu_s$$

Example 4.2 ([1], SP. 6/2) Determine the range of values which the mass m_o may have so that the 100 kg block shown in the figure will neither start moving up the plane nor slip down the plane. The coefficient of static friction for the contact surface is 0.30.

Solution: From the setup configuration, if the counterweight is increased too much, the block will start sliding upward. On the contrary, if the counterweight is decreased too much, the block will start sliding downward, lifting the balance upward. The problem asks the maximum and the minimum weight which still neither makes the block sliding upward or downward. This implies the system in question is on impending. Hence the magnitude of the friction force must be $F = F_{\max} = \mu_s N$. The situations will be splitted into two cases; the block starts moving upward and downward, respectively. In both cases, however, forces in the normal direction are the same. The normal force can be determined first by recalling the equilibrium condition in the normal y-direction. From fig. 4.6,

$$[\textstyle\sum F_y = 0] \qquad N - 100g\cos 20 = 0$$

$$N = 922 \;\; \text{N}$$

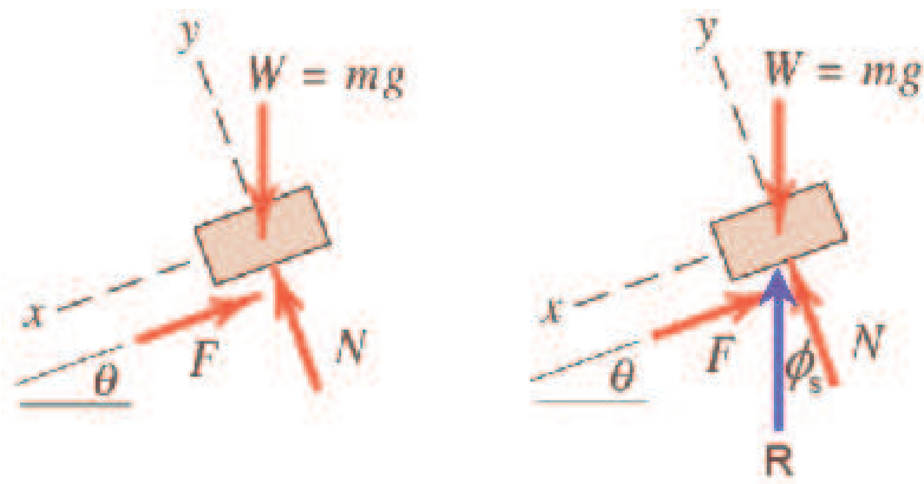


Figure 4.4: Solution to example 4.1 ([1], pp. 346)

Case I: Maximum value of m_o. Block starts moving upward. The resisting friction force pointing downward (fig. 4.6 left):

$[\sum F_x = 0]$
$$m_o g - \mu_s N - 100g \sin 20 = 0$$

$$m_o = 62.4 \ \text{ kg}$$

Case II: Minimum value of m_o. Block starts moving downward. The resisting friction force pointing upward (fig. 4.6 right):

$[\sum F_x = 0]$
$$m_o g + \mu_s N - 100g \sin 20 = 0$$

$$m_o = 6.0 \ \text{ kg}$$

Therefore, mass m_o must be in the range of

$$6.0 \le m_o \le 62.4 \ \text{ kg}$$

so that the block will not start moving.

Example 4.3 ([1], SP. 6/3) Determine the magnitude and direction of the friction force acting on the 100 kg block shown if, first, $P = 500$ N and, second, $P = 100$ N. The coefficient of static friction is 0.20, and the coefficient of kinetic friction is 0.17. The force are applied with the block initially at rest.

Solution: After reading the problem statement, the motion status of the block cannot be deduced. The problem falls in the unknown status case and the

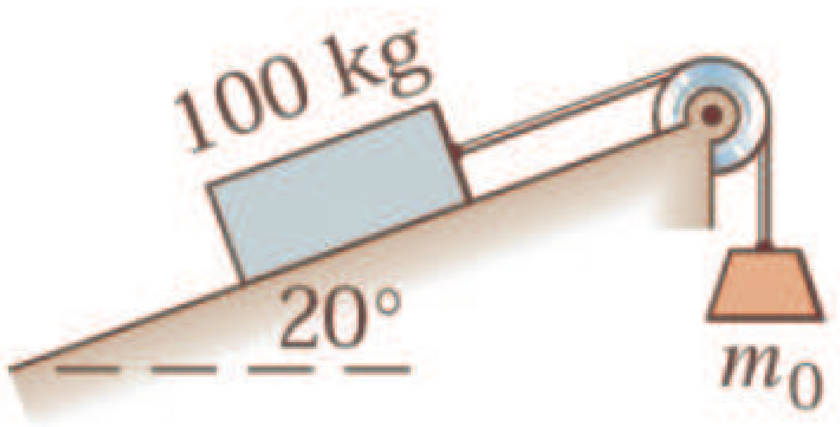


Figure 4.5: Example 4.2 ([1], pp. 346)

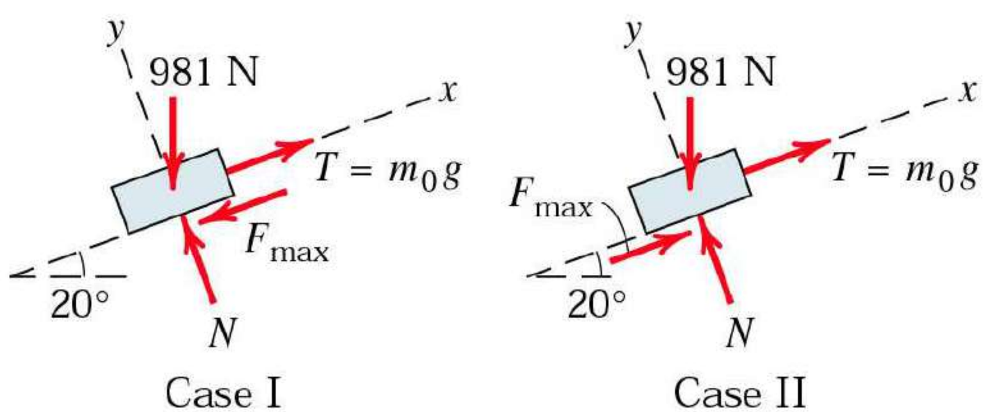


Figure 4.6: Solution to example 4.2 ([1], pp. 346)

approach is to assume the system is in static equilibrium condition. Additionally, it is not known whether the block is likely to move upward or downward the incline. Since this information is not known, it will be assumed as well.

$\underline{P = 500 \ \text{N}}$ Compared to the case of $P = 100$ N, the tendency of the block's motion is assumed to be upward. Therefore the friction direction is assumed downward. After drawing the free body diagram as depicted in fig. 4.8, the equilibrium conditions are applied:

$$[\sum F_y = 0] \qquad N - 500 \sin 20 - 100g \cos 20 = 0$$

$$N = 1092.85 \ \text{N}$$

$$[\sum F_x = 0] \qquad 500 \cos 20 - F - 100g \sin 20 = 0$$

$$F = 134.3 \ \text{N}$$

The maximum supportable friction force is

$$F_{\text{max}} = \mu_s N = 0.2 \times 1092.85 = 218.6 \ \text{N}$$

The required friction force for the equilibrium condition is less than the maximum value. Therefore the friction force is 134.3 N downward and the block is still at rest.

$\underline{P = 100 \ \text{N}}$ Compared to the case of $P = 500$ N, the tendency of the block's motion is assumed to be downward. Therefore the friction direction is assumed upward. After drawing the free body diagram as shown in fig. 4.8, the equilibrium conditions are applied:

$$[\sum F_y = 0] \qquad N - 100 \sin 20 - 100g \cos 20 = 0$$

$$N = 956.04 \ \text{N}$$

$$[\sum F_x = 0] \qquad F + 100 \cos 20 - 100g \sin 20 = 0$$

$$F = 241.55 \ \text{N}$$

The maximum supportable friction force is

$$F_{\text{max}} = \mu_s N = 0.2 \times 956.04 = 191.21 \ \text{N}$$

The required friction force for the equilibrium condition is greater than the maximum value. Therefore this required friction force cannot be fulfilled. The equilibrium assumption is thus invalid and the block is moving downward instead.

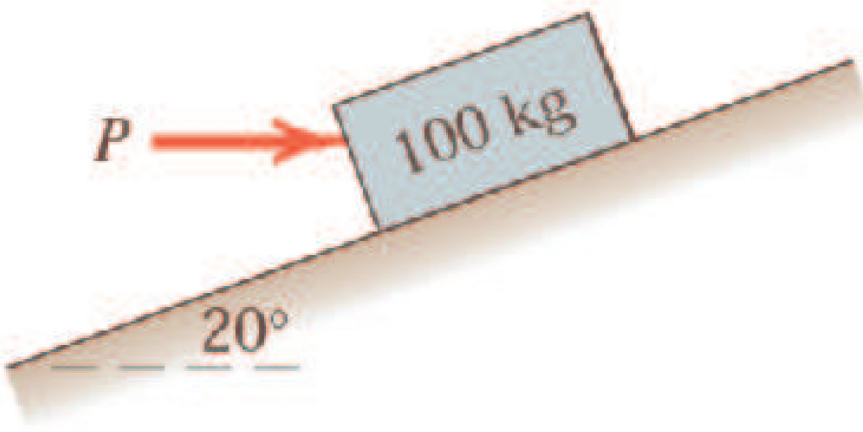


Figure 4.7: Example 4.3 ([1], pp. 347)

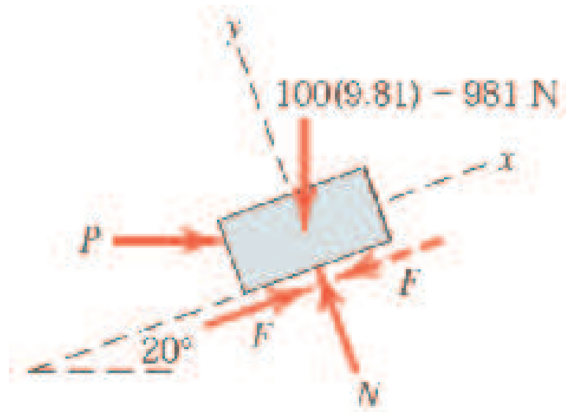


Figure 4.8: Solution to example 4.3 ([1], pp. 347)

The friction force then becomes the kinetic friction force in the upward direction of which its magnitude is

$$F = \mu_k N = 0.17 \times 956.04 = 162.5 \ \ \text{N}$$

Example 4.4 ([1], SP. 6/4) The homogeneous rectangular block of mass m, width b, and height H is placed on the horizontal surface and subject to a horizontal force P which moves the block along the surface with a constant velocity. The coefficient of kinetic friction between the block and the surface is μ_k. Determine

(a) the greatest value that h may have so that the block will slide without tipping over and
(b) the location of a point C on the bottom face of the block through which the resultant of the friction and normal force acts if $h = H/2$.

Solution: In the first question, as the point of application of the applied force P increases, the block will be more likely about to tip over. When the block is about to tip over, the resultant supporting force from the ground must be acting through the last contact point at the far corner of the block, labeled as point A in fig. 4.10. For this problem, the block is being moved with a constant velocity. Therefore the developed friction is the kinetic friction force. From the friction cone viewpoint, the resultant force R must be such that its direction is forming the angle

$$\theta = \tan^{-1} \mu_k$$

with the normal direction. Since the block is in equilibrium (moving with constant velocity), all three forces acting on the block must meet at the common point denoted as B in the figure. The location of this point will dictate the height h of the applied force P. Because the ground reaction force must pass through point A and B, the following geometric relationship can be written:

$$b/2h = \tan\theta = \mu_k$$

Hence the greatest value of h is determined as

$$h = b/2\mu_k$$

For the second question, the height of the point of application is fixed to be $h = H/2$. Nevertheless, the system is still in equilibrium and hence three forces acting on the block must also meet at the common point. In this case the point is known; it is the center of gravity point G. Since the ground reaction force must pass through this point and its direction must be such that

$$\theta = \tan^{-1} \mu_k$$

the location of the point C on the bottom face through which the resultant reaction force acts must satisfy the following relationship:

$$x/\left(H/2\right) = \tan\theta = \mu_k$$

In other words, the location of the point C from the middle vertical line of the block is

$$x = \mu_k H/2$$

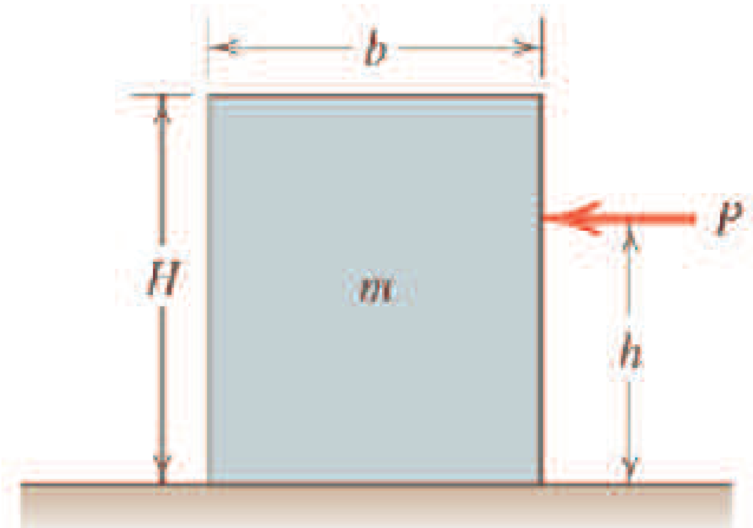


Figure 4.9: Example 4.4 ([1], pp. 348)

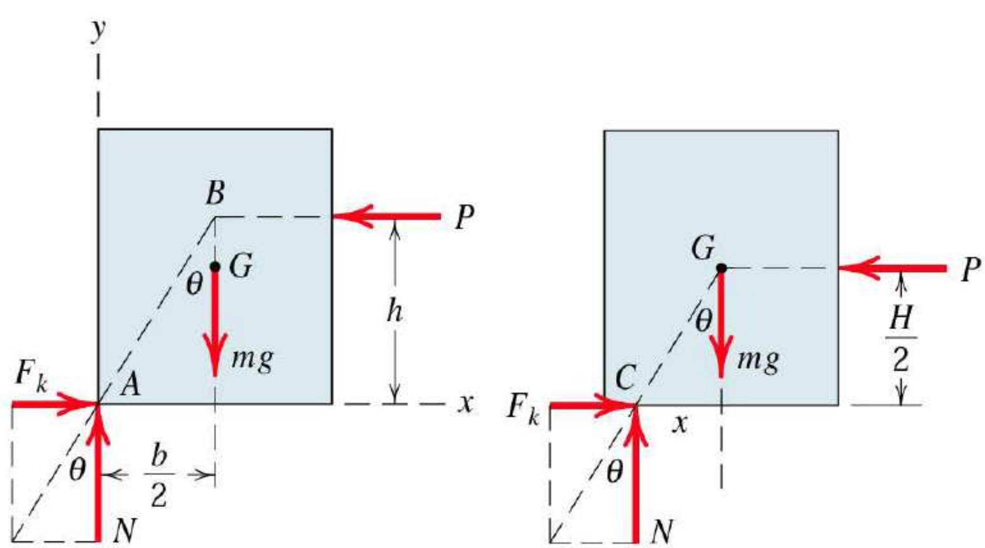


Figure 4.10: Solution to example 4.4 ([1], pp. 348)

Example 4.5 ([1], SP. 6/5) The three flat blocks are positioned on the 30° incline as shown, and a force P parallel to the incline is applied to the middle block. The upper block is prevented from moving by a wire which attaches it to the fixed support. The coefficient of static friction for each of the three pairs of mating surfaces is shown. Determine the maximum value which P may have before any slipping takes place.

Solution: From the arrangement of the blocks, the 30 kg block cannot be moved. Only either the 50 or the 40 kg block, or together both blocks can be moved. However the applied force P acts at the 50 kg block, it is unlikely that the 40 kg block will move alone. Therefore there will be only 2 possible cases: either the 50 kg block alone or the 50 and 40 kg blocks will move together as one unit.

After drawing the free body diagram shown in fig. 4.12, the equilibrium condition in the normal direction of each block requires

$$[\sum F_y = 0] \qquad N_1 - 30g\cos 30 = 0, \quad N_1 = 254.87 \text{ N}$$

$$N_2 - N_1 - 50g\cos 30 = 0, \quad N_2 = 679.66 \text{ N}$$

$$N_3 - N_2 - 40g\cos 30 = 0, \quad N_3 = 1019.5 \text{ N}$$

The maximum friction forces possible developed at each mating surface therefore are

$$\begin{aligned} F_{1\max} &= \mu_s N_1 = 76.46 \text{ N} \\ F_{2\max} &= \mu_s N_2 = 271.86 \text{ N} \\ F_{3\max} &= \mu_s N_3 = 458.8 \text{ N} \end{aligned}$$

The normal force will next be used to determine the friction forces right before any slippage as follow.

<u>50 kg-block tends to move alone</u> This implies F_1 and F_2 must be $F_{1\max}$ and $F_{2\max}$. The block will not slip had either one alone reaches the maximum value. Consider the 40 kg-block.

$$[\sum F_x = 0] \qquad F_2 - F_3 + 40g\sin 30 = 0$$

$$F_3 = 468.06 \text{ N} \quad > F_{3\max}$$

which is not possible. Therefore the 40 kg-block cannot stay still and the requirement of no slippage cannot be satisfied. Hence this case will not happen.

<u>50 and 40 kg-block tend to move together</u> If this would happen, F_1 and F_3 must be $F_{1\max}$ and $F_{3\max}$. Both blocks as an integral unit will not slip had either

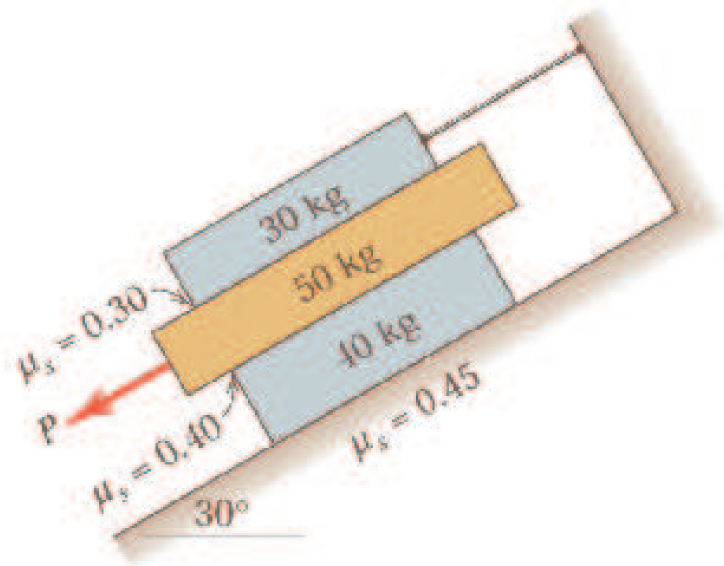


Figure 4.11: Example 4.5 ([1], pp. 349)

one alone reaches the maximum value. Consider the 40 kg-block.

$$[\sum F_x = 0] \qquad F_2 - F_3 + 40g \sin 30 = 0$$

$$F_2 = 262.6 \text{ N} \quad < F_{2\max}$$

which is possible. Therefore the 50 and 40 kg-block will not slip relative to each other and be at rest. To determine the maximum value of the applied force P, apply the equilibrium condition at the middle block.

$$[\sum F_x = 0] \qquad P - F_1 - F_2 + 50g \sin 30 = 0$$

$$P = 93.8 \text{ N}$$

Example 4.6 ([1], Prob. 6/6) The light bar is used to support the 50 kg block in its vertical guides. If the coefficient of static friction is 0.30 at the upper and 0.40 at the lower end of the bar, find the friction force acting at each end for $x = 75$ mm. Also find the maximum value of x for which the bar will not slip.

Solution: The most distinct aspect of the system is that the light bar is a two-force member. Therefore, if it is in equilibrium, the reaction forces acted at both ends must be equal, opposite, and acting along the bar's longtitudinal direction, depicted in fig. 4.14. Decomposing the reaction force into the normal

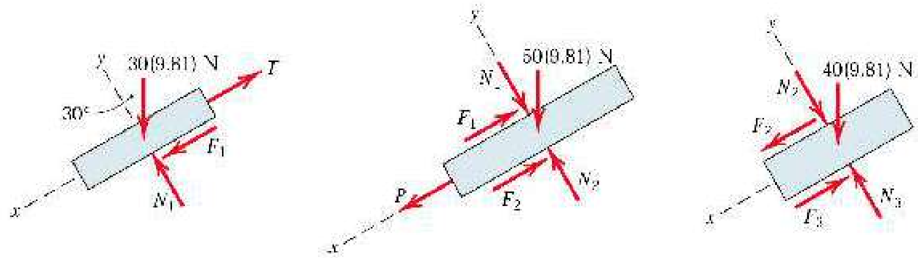


Figure 4.12: Solution to example 4.5 ([1], pp. 349)

and friction force, the normal force must be such that it supports the weight of the 50 kg-block.

$$[\sum F_y = 0] \qquad N - 50g = 0, \quad N = 490.5 \ \text{N}$$

Because the required situation of this problem is static equilibrium, the developed friction must not exceed the maximum value in addition to the two-force member equilibrium condition. The alternative view to this constraint is that the direction of the reaction force must reside in the static friction cone. Hence the angle that the reaction force on each end made with the normal (vertical) direction cannot exceed

$$\begin{aligned} \phi_{sA} &= \tan^{-1}\mu_{sA} = 21.8^\circ \\ \phi_{sB} &= \tan^{-1}\mu_{sB} = 16.7^\circ \end{aligned}$$

Consider the case where the lower end of the bar is displaced by $x = 75$ mm. Direction of the reaction force is along the propped bar. By geometry, the angle the reaction force made with the normal direction is

$$\theta = \sin^{-1}(75/300) = 14.5^\circ < \phi_{sB} < \phi_{sA}$$

The reaction force R is inside the static friction cone and therefore the system is in equilibrium. The friction force can then readily be determined as

$$F = N\tan\theta = 126.6 \ \text{N}$$

To find the maximum value of x for which the bar will not yet slip, one of the reaction forces must be on the static friction cone surface. That reaction force is at the upper end where the coefficient of friction is lower. The lower reaction force will still be inside the static friction cone. At B, the direction of the reaction force, and so the bar, must then be such that

$$x/300 = \sin\phi_{sB}$$

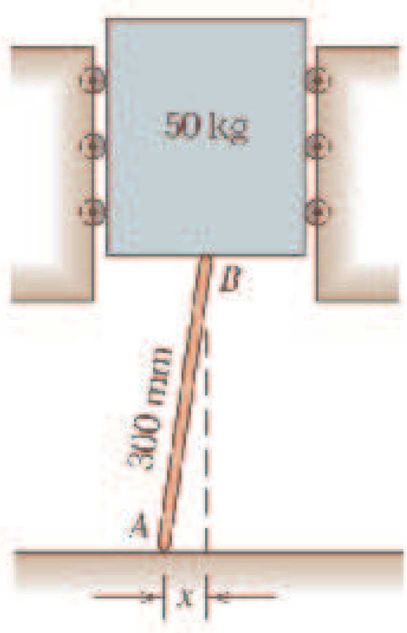


Figure 4.13: Example 4.6 ([1], pp. 351)

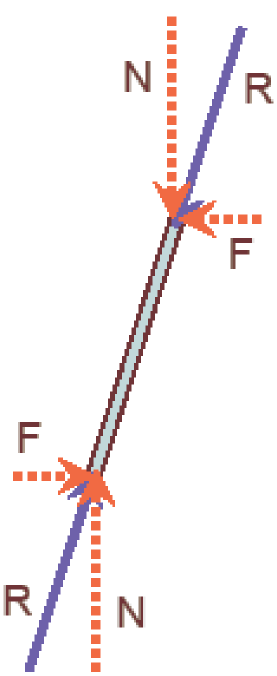


Figure 4.14: Solution to example 4.6

Hence the maximum value of x will be

$$x = 300 \sin 16.7 = 86.2 \ \ \text{mm}$$

Example 4.7 ([1]) Find the tension in the cable and force P that makes the 15 kg lower block

(a) to start sliding downward
(b) to start sliding upward

Solution: This problem is rather simple because the status of the motion is known, i.e. impending motion. Therefore the developed friction force must reach the maximum value. For both cases, the free body diagrams in fig. 4.16 reveal the same force in the normal direction. Imposing the equilibrium condition, the following equations can be written:

$$[\sum F_n = 0] \qquad N_1 = 8g \cos 20 = 73.75 \ \ \text{N}$$

$$N_2 - N_1 - 15g \cos 20 = 0, \quad N_2 = 212 \ \ \text{N}$$

Consequently, the maximum friction forces supportable at each mating pair are

$$\begin{aligned} F_{1\max} &= 0.3N_1 &= 22.12 \ \ \text{N} \\ F_{2\max} &= 0.4N_2 &= 84.81 \ \ \text{N} \end{aligned}$$

a) P pulling down The 15 kg-block impends to slide downward. Because the motion impends, the equilibrium equation of each block along the tangential direction can be set. The applied force P and the cable tension T can then be determined.

$$[\sum F_t = 0] \qquad P - F_{1\max} - F_{2\max} + 15g \sin 20 = 0, \quad P = 56.6 \ \ \text{N}$$

$$F_{1\max} + 8g \sin 20 - T = 0, \quad T = 49 \ \ \text{N}$$

b) P pushing up There are two possible situations. Either the 15 & 8 kg-blocks impend to slide upward together or the 15 kg-block impends to slide upward alone. If both blocks are about to go together, the cable will get slack. This implies $T = 0$ N. Using the equilibrium condition on the 8 kg-block, this requires the resisting friction force of

$$[\sum F_t = 0] \qquad 8g \sin 20 - F_1 = 0, \quad F_1 = 26.84 \ \ \text{N}$$

which is greater than the maximum value of $F_{1\max}$. Consequently, this case cannot happen and therefore the other case that the 15 kg-block impend

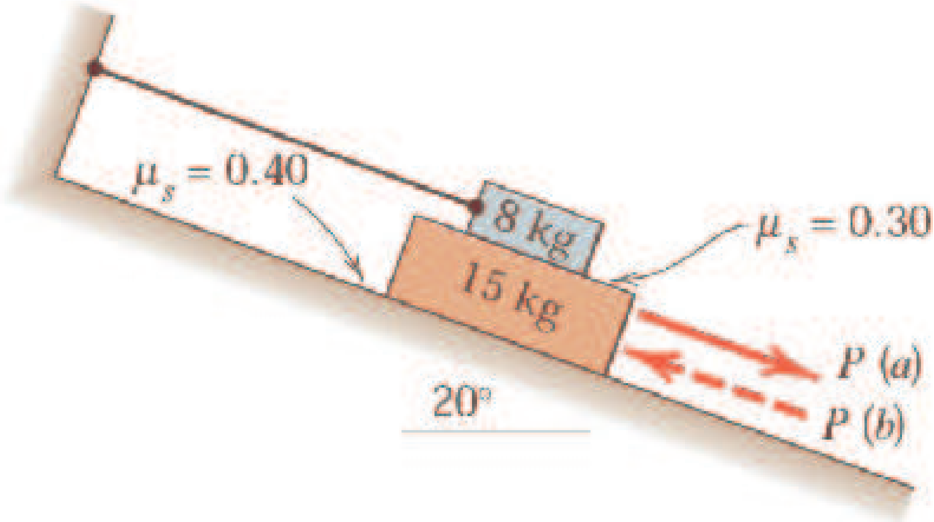


Figure 4.15: Example 4.7 ([1])

to slide upward *alone* will be realized instead. Using the equilibrium conditions along the tangential direction on each block, the applied force P and the cable tension T can be determined.

$$[\sum F_t = 0] \qquad -P + F_{1\max} + F_{2\max} + 15g\sin 20 = 0, \quad P = 157.3 \text{ N}$$

$$-T - F_{1\max} + 8g\sin 20 = 0, \quad T = 4.72 \text{ N}$$

Example 4.8 ([2], Prob. 6/29) The uniform slender rod of mass m and length L is initially at rest in a centered horizontal position on the fixed circular surface of radius $R = 0.6L$. If a force P normal to the bar is gradually applied to its end until the bar begins to slip at the angle $\theta = 20°$, determine the coefficient of static friction.

Solution: The status of the bar is impending motion. From the no slip condition, the distance between the current and the initial contact points of the bar must be equal to the length of the curve traced on the surface, which is

$$[a = \theta r] \qquad a = \left(20 \times \tfrac{\pi}{180}\right) R = \pi R/9$$

After drawing the free body diagram and recognizing the angle α the reaction force made with the normal direction is also the angle of the static friction cone, as shown in fig. 4.18, the coefficient of static friction can be determined as

$$\mu_s = \tan\alpha = \frac{L/2 - \pi R/9}{L/\left(2\tan 20\right)} = 0.211$$

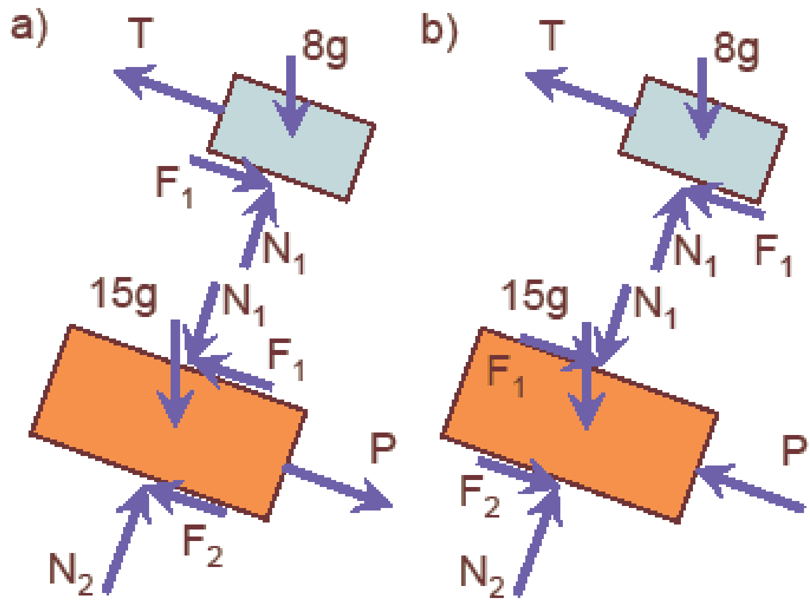


Figure 4.16: Solution to example 4.7

Example 4.9 ([2], Prob. 6/32) The three identical rollers are stacked on a horizontal surface as shown. If the coefficient of static friction μ_s is the same for all pairs of contacting surfaces, find the minimum value of μ_s for which the rollers will not slip.

Solution: After careful consideration of the sketch in fig. 4.19, if the rollers start slipping, the lower rollers tend to roll out at the upper contacts while they tend to slide out at the lower contacts. The top most roller will then start falling down. This described motion is shown in fig. 4.20. The slipping will occur if one or more contacts of the lower rollers impend to slip. To understand the phenomenon, the free body diagram of the lower left roller is drawn as depicted in fig. 4.20. Applying the moment equilibrium at point O,

$$[\sum M_O = 0] \qquad F_A = F_B$$

From the figure, it can be concluded that $N_A < N_B$. Therefore $F_{A\max} < F_{B\max}$. Consequently, F_A will reach the limit value before F_B. This implies slipping will occur first at A and so $F_A = F_{A\max}$. F_B, which may be less than $F_{B\max}$, will be determined by the equilibrium equation.

The roller is recognized as a three-force member. Therefore the line of action of these three forces must intersect at a unique point for the system to be in equilibrium. The only possible meeting point is B. In addition, since the reaction

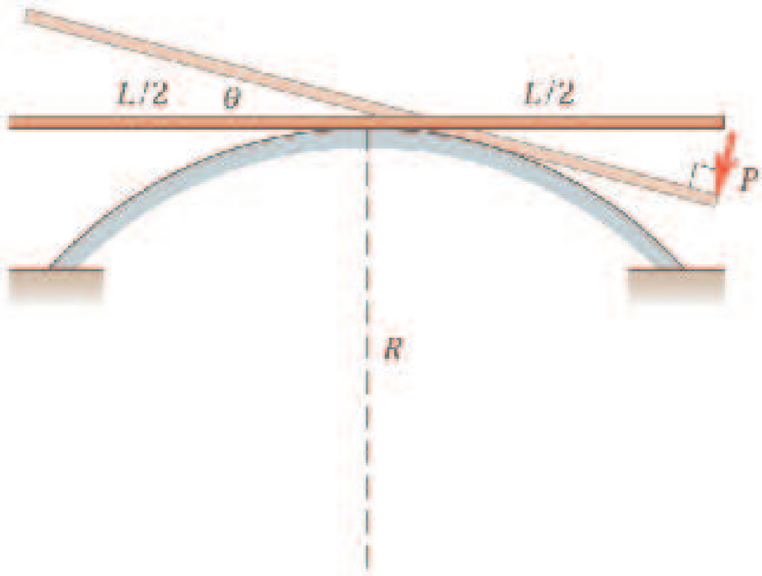


Figure 4.17: Example 4.8 ([2], pp. 362)

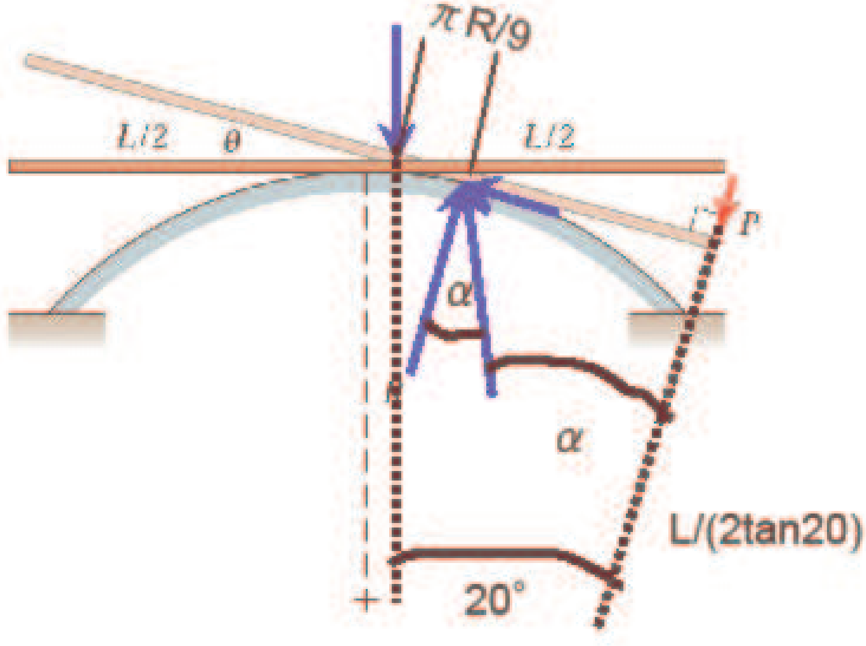


Figure 4.18: Solution to example 4.8

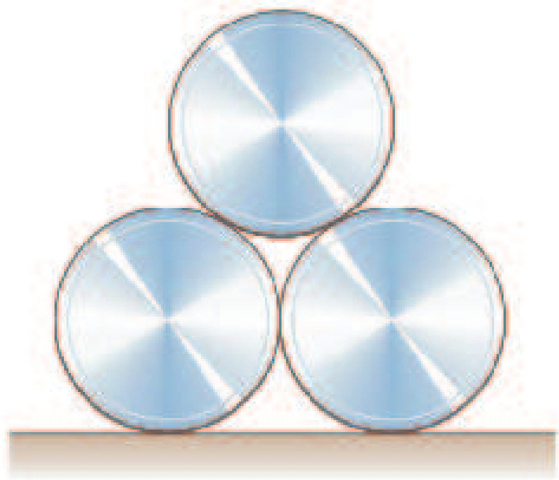


Figure 4.19: Example 4.9 ([2], pp. 363)

force at A reaches the limit, R_A must be making the angle $\tan^{-1}\mu_s$ with N_A. From the geometry of the triangle OAB sketched in fig. **??**, it can be stated that

$$\tan^{-1}\mu_s = 15^\circ$$

As a result, the minimum value of μ_s to guarantee the stacked rollers to retain their equilibrium is

$$\mu_s = \tan 15 = 0.268$$

Example 4.10 ([2], Prob. 6/41) The industrial truck is used to move the solid 1200 kg roll of paper up the 30° incline. If the coefficients of static and kinetic friction between the roll and the vertical barrier of the truck and between the roll and the incline are both 0.40, compute the required tractive force P between the tires of the truck and the horizontal surface.

Solution: The moving paper roll is making contacts with the vertical barrier of the truck and the incline. This problem is difficult because it is not clear from the statement how the roll is moving. After drawing the free body diagram of the roll shown in fig. 4.22, three possibilities are listed.

1. A and B both slip
2. Only B slips
3. Only A slips

Each case has been investigated with the assumption of dynamical equilibrium. After the calculation, only the last case — i.e. the roller slips at A —

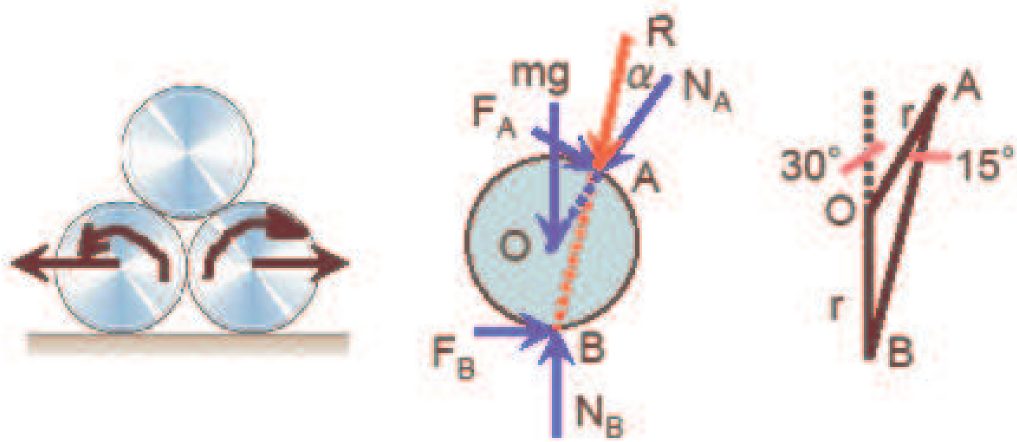


Figure 4.20: Solution to example 4.9

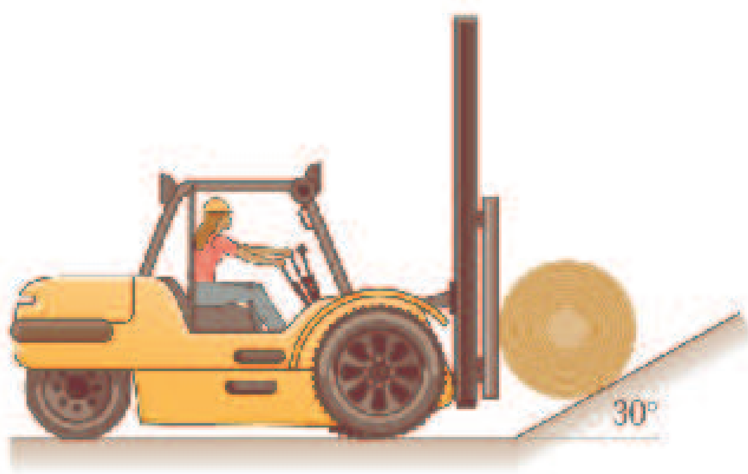


Figure 4.21: Example 4.10 ([2], pp. 365)

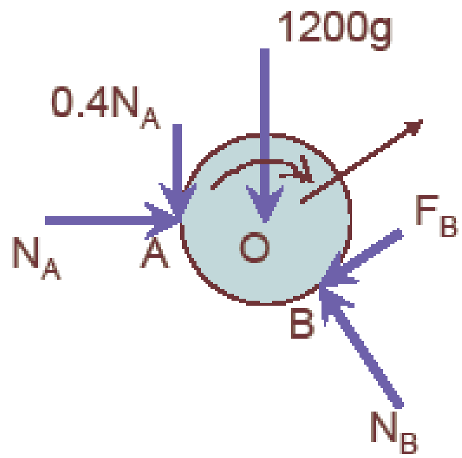


Figure 4.22: Solution to example 4.10

is viable. For this case, because the roll slips at A, the kinetic friction force is determined as

$$F_A = 0.4N_A$$

Applying the equilibrium conditions to the roller, other unknown forces can be determined as follow:

$[\sum M_O = 0]$ $$F_B = F_A = 0.4N_A$$

$[\sum F_x = 0]$ $$N_A - F_B \cos 30 - N_B \sin 30 = 0, \quad N_B = 1.307N_A$$

$[\sum F_y = 0]$ $$-0.4N_A - 1200g - F_B \sin 30 + N_B \cos 30 = 0$$

$$N_A = 22.1 \text{ kN}, \quad N_B = 28.9 \text{ kN}, \quad F_B = 8853 \text{ N} < 0.4N_B$$

which is an indication that the required friction force of F_B is supportable. By observation of the equilibrium truck, the required tractive force P must balance the normal force N_A. Therefore,

$$P = 22.1 \text{ kN} \quad \rightarrow$$

Chapter 5

Distributed Forces

5.1 Introduction

In reality, forces are applied over a region of which often its area is negligible. When forces are applied over a region whose dimensions are not negligible compared with other pertinent dimensions, we must account for the actual manner in which the force is distributed by summing up the effects of the distributed force over the region. For this purpose, we need to know the intensity of the force at any location and we will use the integration to determine their total effect. Figure 5.1 are common engineering examples of the distributed forces.

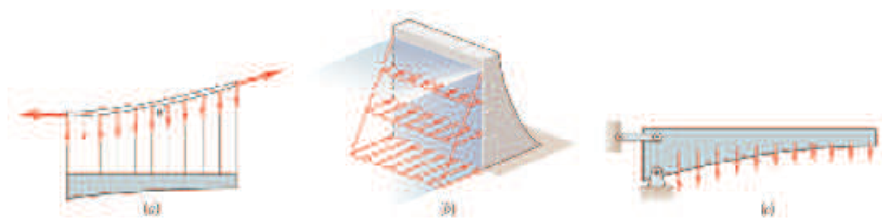


Figure 5.1: Some examples of the distributed forces ([1], pp. 237)

5.2 Center of Mass

In this section, we study the method to determine the location of the center of gravity, the center of mass, and the center of volume. The principle of moment, i.e.

"The sum of the moments is equal to the moment of the sum."

is basically employed to determine these locations.

5.2.1 Center of Gravity

The center of gravity (C.G.) is the point where the resultant gravitational force **W** acts. It is determined from the principle of moment:

"Moment of the resultant gravitational force **W** about any axis is equal to the sum of the moments about the same axis of the gravitational force $d\mathbf{W}$ acting on all particles of the body."

Most of the objects subject to the earth gravity can be safely assumed that the intensity of the earth's force field over the body is uniform. The other reasonable assumption is that the field of force due to the gravitational attraction is parallel. Applying these assumptions to the principle of moment simplifies the derivation of the C.G. because the cross product degenerates to a simple multiplication.

The position of the C.G. is then

$$\mathbf{r}_{\mathrm{CG}} = \frac{\int \mathbf{r}\, dW}{\int dW} \tag{5.1}$$

where $\mathbf{r}$ is the position vector of the particle corresponding to the gravitational force dW. $\mathbf{r}_{\mathrm{CG}}$ is the position vector of the C.G. Both may be described in any coordinate frame such as the rectangular coordinate frame, for which the case $\mathbf{r}$ can be written as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

5.2.2 Center of Mass

The center of mass (C.M.) is the point where the total mass of the object is visually concentrated. It is again determined from the principle of moment. C.M. is independent of the gravitational effect. It is a unique property of the object, depending on its shape and material.

For simplification in the derivation of the C.M., apply the same assumptions as above; uniform intensity of the earth's force field over the body and parallel field of force due to the gravitational attraction. The position of the C.M. is then

$$\mathbf{r}_{\mathrm{CM}} = \frac{\int \mathbf{r}\, dm}{\int dm} \tag{5.2}$$

where $\mathbf{r}$ is the position vector of the particle corresponding to the infinitesimal mass dm. $\mathbf{r}_{\mathrm{CM}}$ is the position vector of the C.M. Note that C.M. is the same as C.G. had the gravity field be treated as uniform and parallel.

5.2.3 Center of Volume

The center of volume (C.V.) is the point where the total volume of the object is visually concentrated. The principle of moment is also used here to determine the point. C.V. is independent of the density of the material. It is a unique property of the shape, depending on its geometry only.

To simplify the derivation of the C.V., we assume constant intensity of the earth's force field over the body and parallel field of force due to the gravitational attraction. The position of the C.V. is then

$$\mathbf{r}_{\mathrm{CV}} = \frac{\int \mathbf{r}\, dV}{\int dV} \tag{5.3}$$

where $\mathbf{r}$ is the position vector of the particle corresponding to the infinitesimal volume dV. $\mathbf{r}_{\mathrm{CV}}$ is the position vector of the C.V. dV may be further elaborate as $dxdydz$ had the rectangular coordinate frame been used. Note that C.V. will be the same point as C.M. if the density (mass intensity) is uniform over the volume.

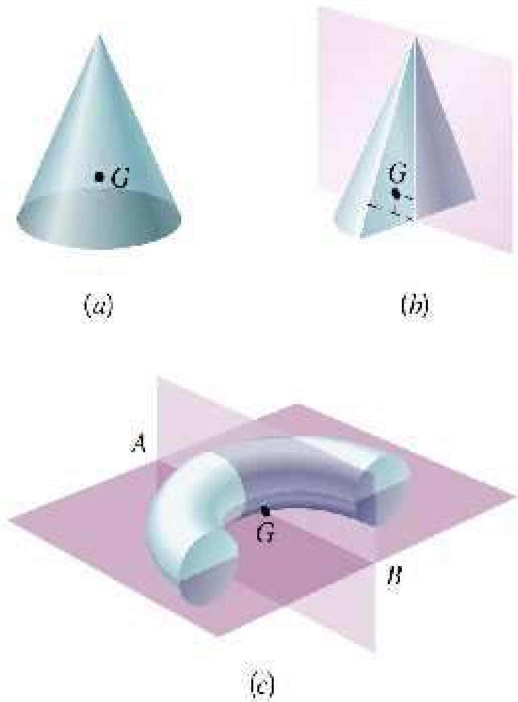


Figure 5.2: Examples of symmetric objects ([1], pp. 239)

Sometimes it is advantageous to recognize the symmetry of the object. This will help reducing the effort in determining the C.V. because there will be some relations among the coordinates. See fig. 5.2 for the examples. Normally, the C.V. will lie on the symmetric features of the object. For example, the cone in fig. 5.2a has the axis of symmetry. So the C.V. will be on the axis. Half of the cone has the plane of symmetry as shown in fig. 5.2b. Hence the C.V. must be somewhere on the plane. Or the half ring shape object has two planes of symmetry. Therefore the C.V. must be on the intersecting line of these two planes, as shown in fig. 5.2c. Acknowledging these facts eliminate the need to determine some centroidal coordinates.

If the body is made from the homogeneous material, the above discussion can be extended to the determination of the C.M. That is, it will lie on the symmetric features of the object as well. In fact, it will be the same point as the C.V.

5.3 Centroids (Line, Area, Volume)

Another name of the C.V. is the centroid. It is purely the geometrical (shape) property of the body since any reference to its mass properties has disappeared. We can divide the determination of the centroid into the case for the line, area, and volume.

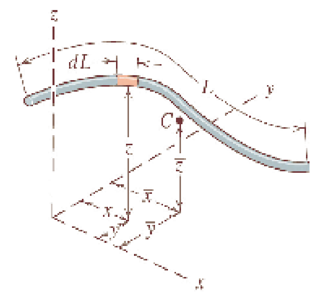


Figure 5.3: Determination of the line centroid ([1], pp. 240)

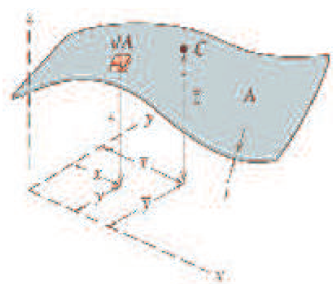


Figure 5.4: Determination of the area centroid ([1], pp. 240)

Line We assume the cross sectional area is constant over the length. The centroid of the line is determined as

$$\mathbf{r}_{\mathrm{C}} = \frac{\int \mathbf{r}_c \, dL}{\int dL} \tag{5.4}$$

Note that the centroid may not always be on the line. See fig. 5.3.

Area We assume the thickness is constant over the entire area. The centroid of the area is determined as

$$\mathbf{r}_{\mathrm{C}} = \frac{\int \mathbf{r}_c \, dA}{\int dA} \tag{5.5}$$

$\int \mathbf{r}_c \, dA$ is called the first moment of area. Note that the centroid may not always be on the surface. See fig. 5.4.

Volume As studied in the previous section, the centroid of the volume can be determined as

$$\mathbf{r}_{\mathrm{C}} = \frac{\int \mathbf{r}_c \, dV}{\int dV} \tag{5.6}$$

The centroid does not need to always be inside the volume.

Guidelines Here are some guidelines in choosing the differential elements and setting up the integrals to determine the centroid.

1. *Order of element* Usually, we prefer first order differential element to higher order to avoid multiple integration. For examaple, we can determine the area of the shape in fig. 5.5a in two different ways:

   $$A = \int l dy, \quad l = l\left(y\right)$$

   or

   $$A = \iint dxdy$$

   However, the first method is preferable. Another example is to determine the volume of the object in fig. 5.5b.

   $$V = \int \pi r^2 dy, \quad r = r\left(y\right)$$

   or

   $$V = \iiint dxdydz$$

   However, the first method is preferable because it is more difficult in this case to determine the limit of integration of each variable.

2. *Continuity* We prefer choosing the element that can be integrated in one continuous operation to cover the whole object. The area in fig. 5.6 can be determined by integrating with respect to x or y. Yet we recommend the approach in fig. 5.6b where the area can be determined by evaluating just one integral expression. The difficulty is hidden in expressing the length l as a function of the integration variable y.

3. *Higher order terms* Higher order terms may be dropped compared with lower order terms. This can be done without introducing any error because in the limit the area or volume of the differential element of higher order terms becomes virtually zero. Figure 5.7 illustrates the evaluation of the area. With too much worry, one may be tempted to determine the area of the given shape by taking into account the small area on top of each column with the triangular area as well:

   $$A = \int ydx + \int \frac{1}{2} dxdy = \int \left(y + \frac{1}{2}dy\right) dx = \int ydx$$

   Since $\frac{1}{2}dy$ is far less than y, we can safely ignore it. Hence the expression for the area considering the infinitesimal triangles eventually becomes the same as if they have not been included.

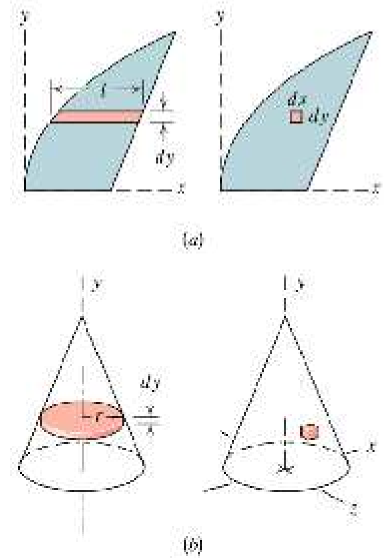


Figure 5.5: Select the differential element, of which the volume is known, to be as large as possible ([1], pp. 241)

4. Choice of coordinates It is advised to choose the coordinate system describing the area or volume as appropriate for the particular shape of the object. If the body is of rectangular shape, we should use rectangular coordinates. However, if its shape is circular, we should employ polar coordinates instead. If the boundary is described with rectangular coordinates, as shown in fig. 5.8a for example, the area should be determined from

$$A = x_o y_o - \int_0^{y_o} x dy$$

On the other hand, if the boundary is explained using the polar coordinates, as in fig. 5.8b, the area should be calculated as

$$A = \int_0^{\theta_o} \frac{1}{2} r^2 d\theta$$

5. Centroidal coordinate of the element Since we usually use the *coordinate of the centroid of the element* for the moment arm in setting up the moment of the differential element, it is recommended to divide the object in the manner that the centroid of these differential elements can be expressed easily in terms of the selected coordinates. It may be recognized immediately that the centroid of the differential element in fig. 5.9a is at the middle point of the strip. For the case of fig. 5.9b, we may look up the standard table of moment of inertia and centroid for the location of the centroid of the differential element.

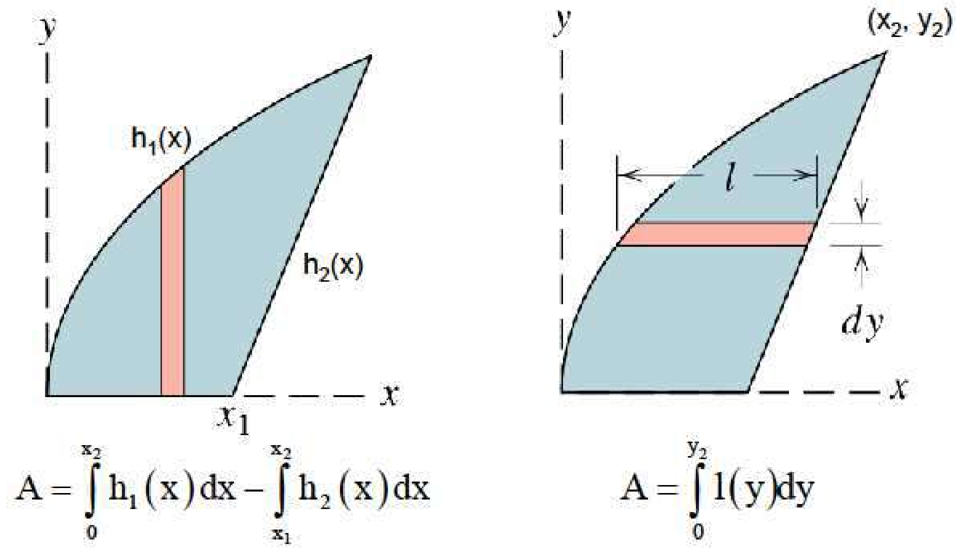


Figure 5.6: If possible, divide the object in the manner that can be integrated in one continuous operation to cover the whole body ([1], pp. 241)

Example 5.1 ([1], SP. 5/3) Locate the centroid of the area of a circular sector with respect to its vertex.

Solution: There are several ways to determine the centroid depending on how to divide the area into differential elements. Here, two solutions are presented.

Infinitesimal concentric ring The area is divided into infinitesimal concentric rings as shown in the left of fig. 5.11. In this given area, we observe the symmetry around the horizontal line. Therefore, it can be conclude that the y-coordinate of the centroid is

$$\overline{Y} = 0$$

Now the x-coordinate of the centroid, $\overline{X}$, will be determined. The infinitesimal area of a concentric ring located at r_o, as highlighted in red, is

$$dA = 2\alpha r_o dr_o$$

The centroid of this ring can be determined as a separted problem of finding the centroid of the partial circular ring of radius r_o and of angle 2α. As a result, this ring has the centroid being located at

$$x_c = \frac{r_o \sin\alpha}{\alpha}$$

Recall the equation to determine the centroid for x-coordinate. Therefore,

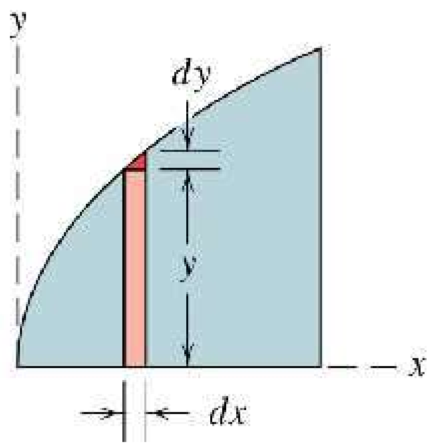


Figure 5.7: The higher order terms can safely be omitted from the integrand ([1], pp. 241)

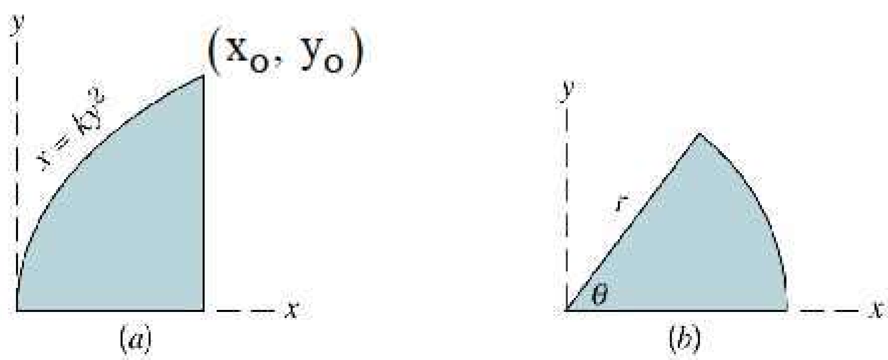


Figure 5.8: Select the coordinates that suit to the shape and boundaries of the object ([1], pp. 242)

$\left[\overline{X}A = \int x_c dA\right]$ $\quad\quad\quad \overline{X}\int_0^r 2\alpha r_o dr_o = \int_0^r \frac{r_o \sin\alpha}{\alpha} 2\alpha r_o dr_o$

$$\overline{X} = \frac{2r\sin\alpha}{3\alpha}$$

<u>Infinitesimal sector</u> Here the area is divided into infinitesimal sectors as shown in the right of fig. 5.11. Again, from the observation, $\overline{Y} = 0$. Only the x-coordinate of the centroid, $\overline{X}$, will be determined. The infinitesimal area of a sector at the angle θ, as highlighted in red, is

$$dA = \frac{1}{2}r\,(rd\theta)$$

The centroid of this ring can be determined as a separted problem of finding the centroid of the infinitesimal sector of radius r located at the angle θ. As a result,

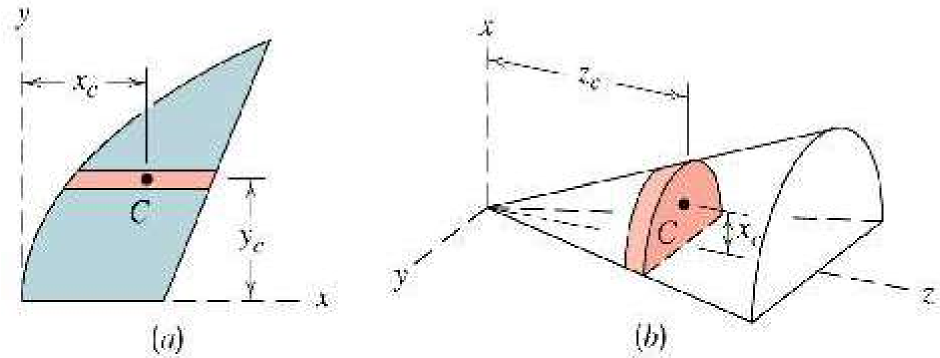


Figure 5.9: Divide the object in the manner that the centroid be expressed easily in terms of the selected coordinates ([1], pp. 242)

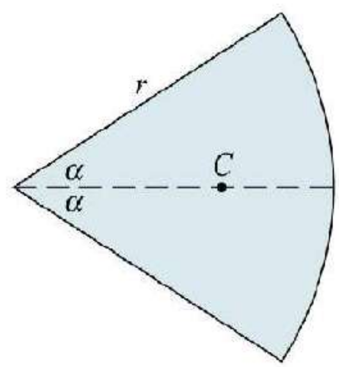


Figure 5.10: Example 5.1 ([1], pp. 245)

the sector has the centroid being located at

$$x_c = \left(\frac{2}{3}r\right)\cos\theta$$

Recall the equation to determine the centroid for x-coordinate. Therefore,

$$\left[\overline{X}A = \int x_c dA\right] \qquad \overline{X}\int_{-\alpha}^{\alpha}\tfrac{1}{2}r^2 d\theta = \int_{-\alpha}^{\alpha}\tfrac{2}{3}r\cos\theta \times \tfrac{1}{2}r^2 d\theta$$

$$\overline{X} = \frac{2r\sin\alpha}{3\alpha}$$

Example 5.2 ([2], Prob. 5/23) Locate the centroid of the area shown in the figure by direct integration.

Solution: It can be observed that the given area is symmetric about the

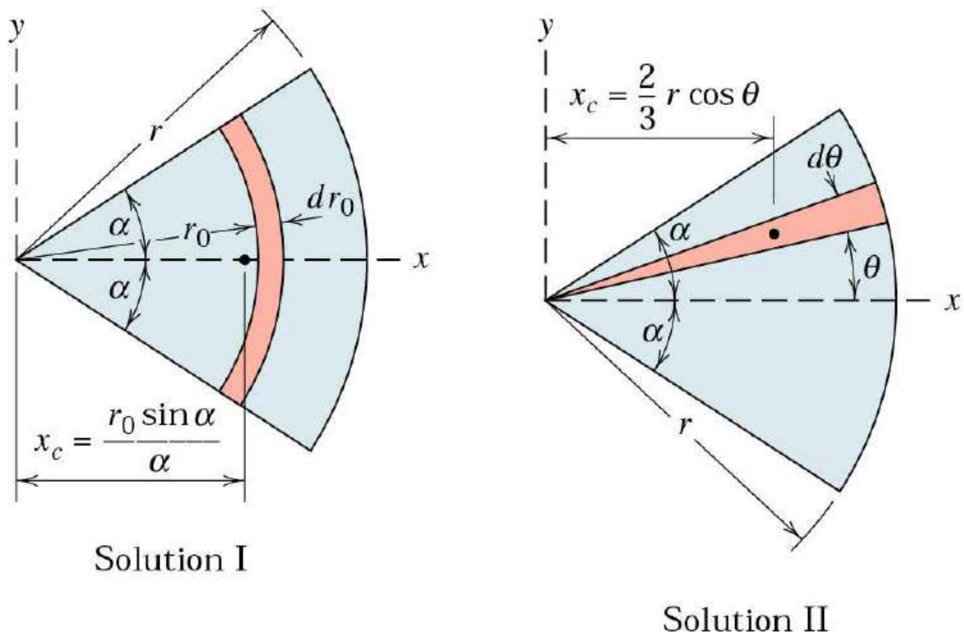


Figure 5.11: Solution to example 5.1 ([1], pp. 245)

line $y = a - x$. Therefore the centroid will lie on this line. This implies

$$\overline{Y} = a - \overline{X}$$

The shaded area can be viewed as the summation of the horizontal strips. Each strip is built from subtracting the horizontal strip of the length a with the differential sector located at the corresponding angle θ. These strips will be evaluated through the whole range of θ varying from 0 to $\pi/2$.

Each infinitesimal area is then

$$dA = ady - \frac{1}{2}a\,(ad\theta)$$

Transform the y- to θ- coordinate to perform the integration under the same variable. We have the following relationship:

$$y = a - a\cos\theta$$

$$dy = a\sin\theta d\theta$$

The x-coordinate centroid of these differential elements can be observed as

$$x_{c1} = a/2$$

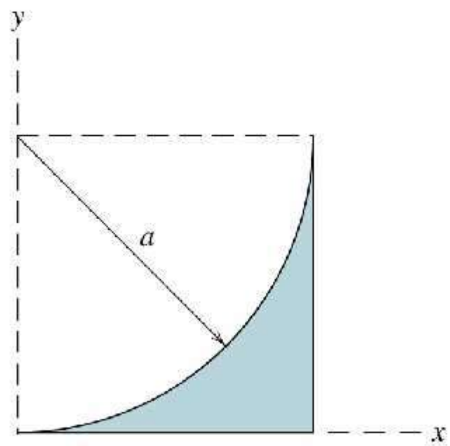


Figure 5.12: Example 5.2 ([2], pp. 258)

$$x_{c2} = \frac{2}{3} a \sin\theta$$

for the horizontal and the sector strips, respectively. Now it is ready to determine the x-coordinate centroid of this area.

$$\left[\overline{X}A = \int x_c dA\right]$$

$$\overline{X} \int_0^{\pi/2} a^2 \left(\sin\theta - \frac{1}{2}\right) d\theta = \int_0^{\pi/2} \frac{a}{2} \times a^2 \sin\theta d\theta - \int_0^{\pi/2} \left(\frac{2}{3} a \sin\theta\right) \times \frac{1}{2} a^2 d\theta$$

$$\overline{X} = \frac{2a}{3(4-\pi)}$$

Recall the centroid location on the line $y = a - x$. Hence, the y-coordinate centroid of the area is

$$\overline{Y} = a - \overline{X} = \frac{(10 - 3\pi)\, a}{3(4-\pi)}$$

Example 5.3 ([2], Prob. 5/35) Determine the coordinates of the centroid of the volume obtained by revolving the shaded area about the z-axis through 90° angle.

Solution: This revolved volume is symmetric about the vertical plane making 45° with the x-axis. Therefore $\overline{X} = \overline{Y}$.

The given volume may be dissected as shown in fig. 5.15. The y-coordinate may be chosen as the integration variable varying from 0 to a. The circular edge in the yz plane has the following relation:

$$y^2 + (z-a)^2 = a^2$$

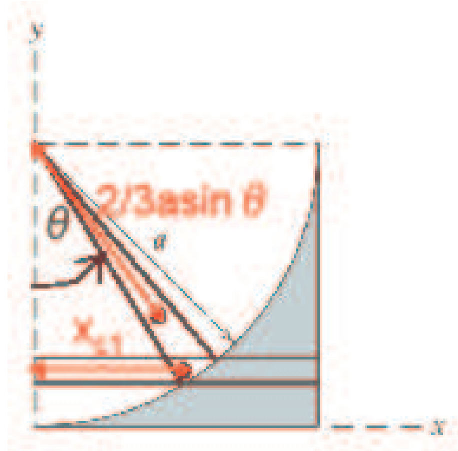


Figure 5.13: Solution to example 5.2

Or in the first quadrant,

$$z = a - \sqrt{a^2 - y^2}$$

Therefore, we can express the differential volume shown in fig. 5.15 in terms of the y-coordinate variable as

$$dV = zdy \times \frac{2\pi y}{4} = \frac{\pi}{2}\left(a - \sqrt{a^2 - y^2}\right) ydy$$

Next is to determine the centroidal coordinates of this differential volume. The z- centroidal coordinate, z_c, is simply half of the height, i.e.

$$z_c = \frac{z}{2} = \frac{a - \sqrt{a^2 - y^2}}{2}$$

The x- centroidal coordinate, x_c, can be determined by using the result of Ex. 5.1. Hence

$$x_c = \frac{y \sin(\pi/4)}{\pi/4} \cos(\pi/4) = 2y/\pi$$

which, by symmetry, is the same as y_c.

Now, it is straightforward to determine the centroidal coordinates of the given volume using the moment equation.

$\left[\overline{X} V = \int x_c dV\right]$

$$\overline{X} \int_0^a \frac{\pi}{2}\left(a - \sqrt{a^2 - y^2}\right) ydy = \int_0^a \frac{2y}{\pi} \times \frac{\pi}{2}\left(a - \sqrt{a^2 - y^2}\right) ydy$$

$$\overline{X} = \left(\frac{4}{\pi} - \frac{3}{4}\right) a = \overline{Y}$$

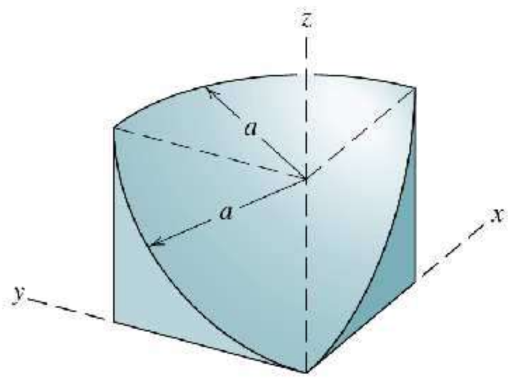


Figure 5.14: Example 5.3 ([2], pp. 261)

$\left[\overline{Z}V = \int z_c dV\right]$

$$\overline{Z}\int_0^a \frac{\pi}{2}\left(a - \sqrt{a^2 - y^2}\right) y dy = \int_0^a \frac{a - \sqrt{a^2 - y^2}}{2} \times \frac{\pi}{2}\left(a - \sqrt{a^2 - y^2}\right) y dy$$

$$\overline{Z} = \frac{a}{4}$$

Example 5.4 ([2], Prob. 5/40) Locate the center of mass G of the steel half ring.

Solution: It is readily seen that the planes of symmetry of the steel half ring is the middle vertical plane and the zero horizontal plane. Therefore the centroid lies on the intersection line of these two planes. Consequently, only the distance $\overline{r}$ is needed to be determined. To do so, the differential volume shaded in red, as shown in fig. 5.17, is selected. If the x- coordinate varies from $-a$ to a, the whole steel half ring is addressed. The other coordinates then need to be expressed as the functions of x variable. From the circular-shape cross section along the ring, we can write

$$x^2 + y^2 = a^2$$

Or

$$y = \sqrt{a^2 - x^2}$$

for the upper half of the circle. The differential volume can now be expressed as

$$dV = \left(2\sqrt{a^2 - x^2}dx\right) \times \pi\,(R - x)$$

Using the result of Ex. 5.1, the centroid of the differential volume is located at

$$r_c = \frac{(R - x)\sin(\pi/2)}{\pi/2} = \frac{2}{\pi}(R - x)$$

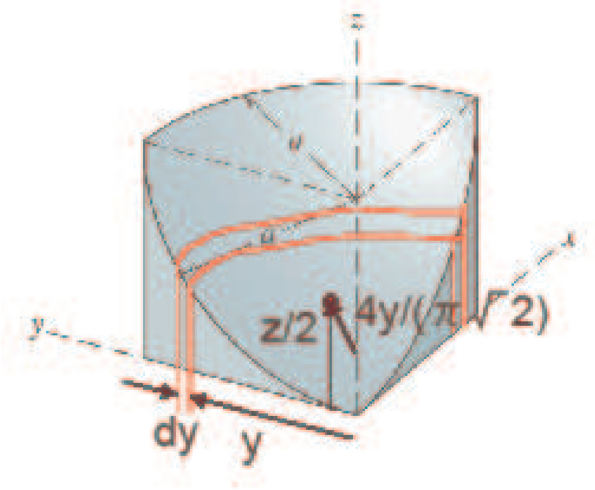


Figure 5.15: Solution to example 5.3

The center of mass G of the steel half ring can now be determined by substituting the above quantities into the moment equation:

$$\left[\bar{r}V = \int r_c dV\right]$$

$$\bar{r}\int_{-a}^{a} 2\pi\left(R-x\right)\sqrt{a^2-x^2}\,dx = \int_{-a}^{a}\frac{2}{\pi}\left(R-x\right)\times 2\pi\left(R-x\right)\sqrt{a^2-x^2}\,dx$$

$$\bar{r} = \frac{a^2+4R^2}{2\pi R}$$

5.4 Composite Bodies and Figures

Figure 5.18 is an example of the complicated object that its centroid cannot be determined easily by analytical integration approach. Fortunately, we may be able to determine it by dividing the original object into sub-parts, for which their centroids can be determined, recognize, or look up more easily. This is the discrete version in finding the centroid compared to the continuous one by dissecting the object into infinitesimal elements. Here we divide the object into smaller ones with known centroids at convenience, as depicted conceptually in fig. 5.18, then the centroid of the complete object may be determined by recalling the principle of moment. However, for this case, the addition of the moment is done discretely using the summation rather than the integration technique.

Summation seems to be more primitive than integration. Therefore we may employ this technique to numerically evaluate the volume and centroid of any

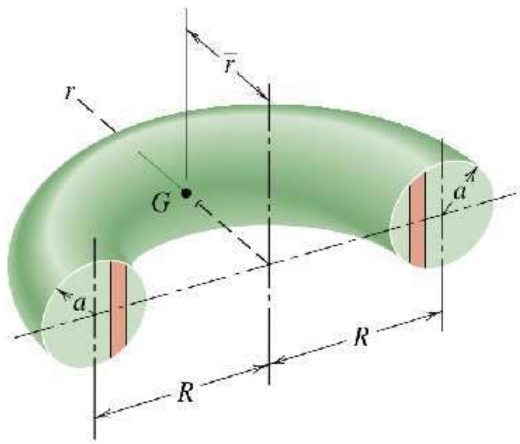


Figure 5.16: Example 5.4 ([2], pp. 262)

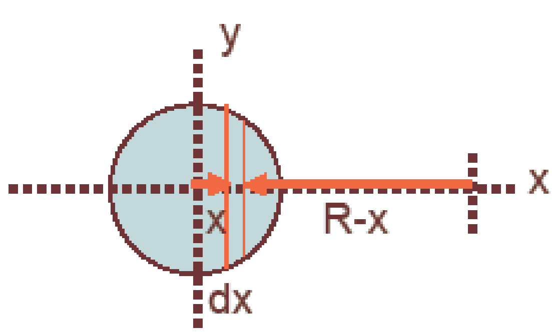


Figure 5.17: Solution to example 5.4

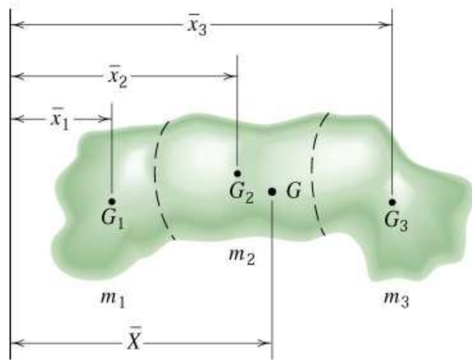


Figure 5.18: The centroid of a complex body may be determined discretely ([1], pp. 256)

object with arbitrary shape, much like the numerical integration techniques performed by mathematical programs to solve the integration that has no analytical solution. Error magnitude of the result may be specified and achieved by modifying the shape and size of the elements. Formulae to determine the coordinates of the center of mass discretely can be expressed as follow:

$$\begin{aligned} \overline{X} &= \frac{\sum_i m_i \overline{x}_i}{\sum_i m_i} \\ \overline{Y} &= \frac{\sum_i m_i \overline{y}_i}{\sum_i m_i} \\ \overline{Z} &= \frac{\sum_i m_i \overline{z}_i}{\sum_i m_i} \end{aligned} \tag{5.7}$$

If the density is uniform throughout the object, the mass m in the above equations may be replaced by the length, l, the area, A, or the volume, V, depending on its geometry. The equations will now determine the coordinates of the centroid discretely.

For the object that has holes or hollows inside its body, they will be accounted by subtracting their moment contribution from the fictitious body with no such features. That is the mass or volume will become negative. Also, it is advised to tabulate the related arguments for the systematic summation purpose. Such arguments are, for example, m_i, $\overline{\mathbf{r}}_{ci}$, and $m_i\overline{\mathbf{r}}_{ci}$.

Example 5.5 ([2], Prob. 5/54) Determine the coordinates of the centroid of the shaded area.

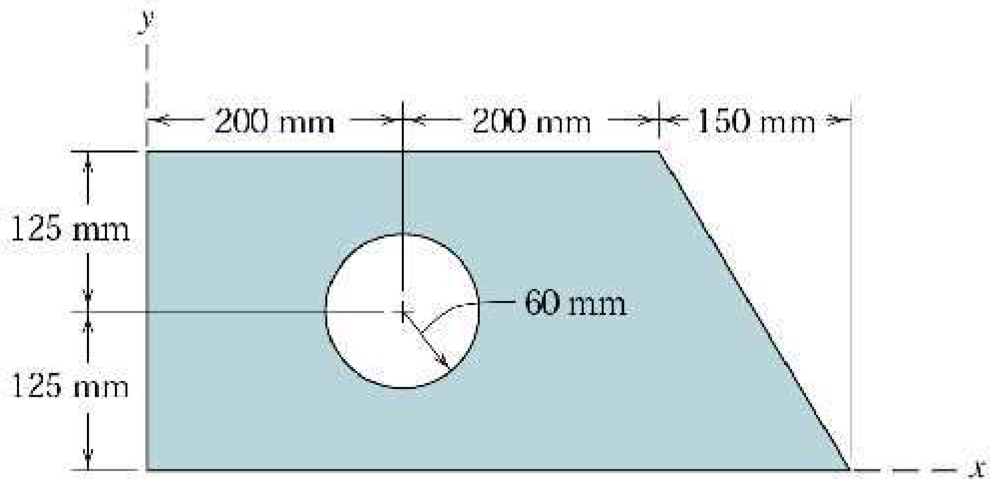


Figure 5.19: Example 5.5 ([2], pp. 269)

Solution: We divide the area into three parts: the rectangle, the triangle, and the negative area circle. Each of these can be determined readily their area and the centroid as shown in Table 5.1 with respect to the given coordinate frame in fig. 5.20.

i	A_i	x_{ci}	y_{ci}	$A_i x_{ci}$	$A_i y_{ci}$
1	0.1	0.2	0.125	0.02	0.0125
2	0.01875	0.45	0.083	8.4375E-3	1.5625E-3
3	-0.0113	0.2	0.125	-2.262E-3	-1.4137E-3

Table 5.1: Table of areas, their moments, and centroids of fig. 5.19

After the table is created, the centroid of the shaded area can be determined directly from eq. (5.7) as

$$\left[\overline{X} = \frac{\sum_i A_i \overline{x}_i}{\sum_i A_i}\right] \qquad \overline{X} = \frac{0.02+8.4375\text{E-}3-2.262\text{E-}3}{0.1+0.01875-0.0113} = 243.6 \ \text{ mm}$$

$$\left[\overline{Y} = \frac{\sum_i A_i \overline{y}_i}{\sum_i A_i}\right] \qquad \overline{Y} = \frac{0.0125+1.5625\text{E-}3-1.4137\text{E-}3}{0.1+0.01875-0.0113} = 117.7 \ \text{ mm}$$

Example 5.6 ([2], Prob. 5/72) A cylindrical container with an extended rectangular back and semicircular ends is all fabricated from the same sheet-metal stock. Calculate the angle α made by the back with the vertical when the container rests in an equilibrium position on a horizontal surface.

Solution: Since the container rests in an equilibrium position, its free body diagram will be drawn and the equilibrium condition will be applied. This will

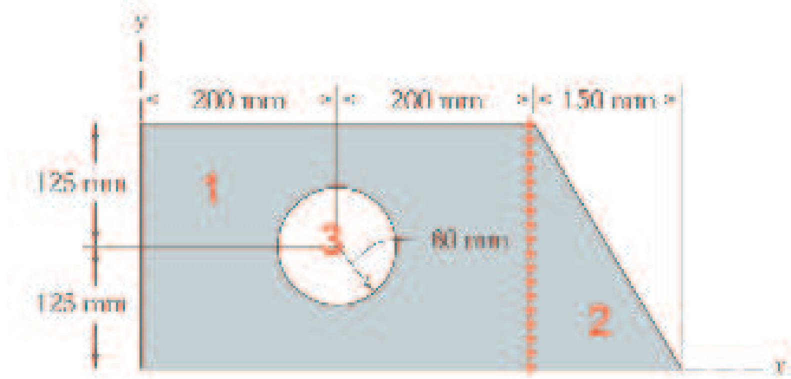


Figure 5.20: Solution to example 5.5

allow us to solve for the required angle α. The force acting on the body would be the ground reaction force and the gravitational force. Hence it is necessary to locate its center of gravity. Since the container is fabricated from the same material, the center of gravity would be the same as the centroid. In addition, the metal sheet used has uniformly 'thin' thickness. Therefore, the centroid of the volume would be the same as the centroid of the area of the sheet.

We divide the container into four parts: the rectangular back, two semi-circular ends, and the half-cylindrical shell. Each of these can be determined readily their area and the centroid as shown in Table 5.2 with respect to the given coordinate frame in fig. 5.22.

i	x_{ci}	y_{ci}	z_{ci}	A_i	$A_i x_{ci}$	$A_i y_{ci}$	$A_i z_{ci}$
1	200	0	100	8E4	16E6	0	8E6
2	200	-150	-300/π	6πE4	12πE6	-9πE6	-18E6
3	0	-150	-200/π	1.125πE4	0	-1.6875πE6	-2.25E6
4	400	-150	-200/π	1.125πE4	4.5πE6	-1.6875πE6	-2.25E6

Table 5.2: Table of areas, their moments, and centroids of fig. 5.21

After the table is created, the centroid can be determined directly from eq. (5.7) as

$$\left[\overline{X} = \frac{\sum_i A_i \overline{x}_i}{\sum_i A_i}\right] \qquad \overline{X} = \frac{16E6+12\pi E6+4.5\pi E6}{8E4+6\pi E4+1.125\pi E4+1.125\pi E4} = 200 \quad \text{mm}$$

$$\left[\overline{Y} = \frac{\sum_i A_i \overline{y}_i}{\sum_i A_i}\right] \qquad \overline{Y} = \frac{-9\pi E6-1.6875\pi E6-1.6875\pi E6}{8E4+6\pi E4+1.125\pi E4+1.125\pi E4} = -114.62 \quad \text{mm}$$

$$\left[\overline{Z} = \frac{\sum_i A_i \overline{z}_i}{\sum_i A_i}\right] \qquad \overline{Z} = \frac{8E6-18E6-2.25E6-2.25E6}{8E4+6\pi E4+1.125\pi E4+1.125\pi E4} = -42.75 \quad \text{mm}$$

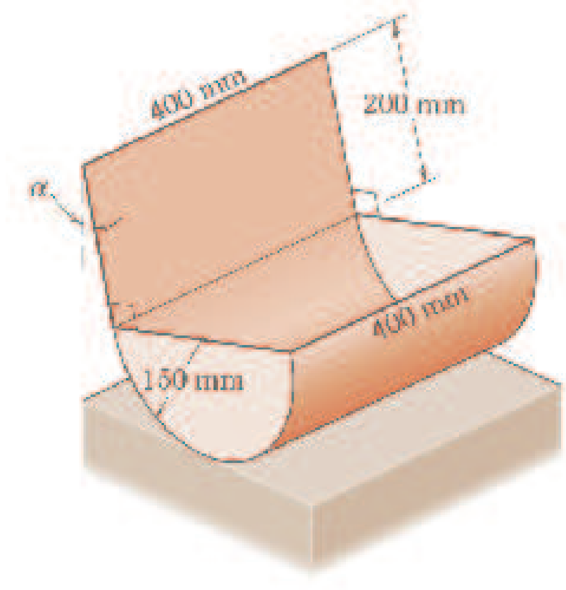


Figure 5.21: Example 5.6 ([2], pp. 274)

These are the x, y, and z coordinates of the center of gravity as well. Looking perpendicular to the yz-plane, the free body diagram along with its coordinate system are displayed in fig. 5.22. There will be the equal and opposite ground reaction force to the gravity force to make the container be in equilibrium. Both forces are in vertical direction and pass through the center of gravity. Since a radius is always perpendicular to the tangent line of the circle, the radius will be the vertical line, coincident with the direction of forces, perpendicular to the ground. If α is the angle made by the back with the vertical (see fig. 5.22), the following geometric relationship can be stated:

$$\alpha = \tan^{-1}\left(\frac{150 - |Y|}{|Z|}\right) = 39.6°$$

5.5 Theorem of Pappus

Several practical objects have their surface or volume created by revolving the planar curve or the planar area about the nonintersecting line in its plane. Pappus theorem is used to determine the area or volume of the revolved object. This is done by dividing the object into infinitesimal circular-arc strips along the axis of revolution. Then the total area or volume may be determined from integrating the infinitesimal area or volume of these strips.

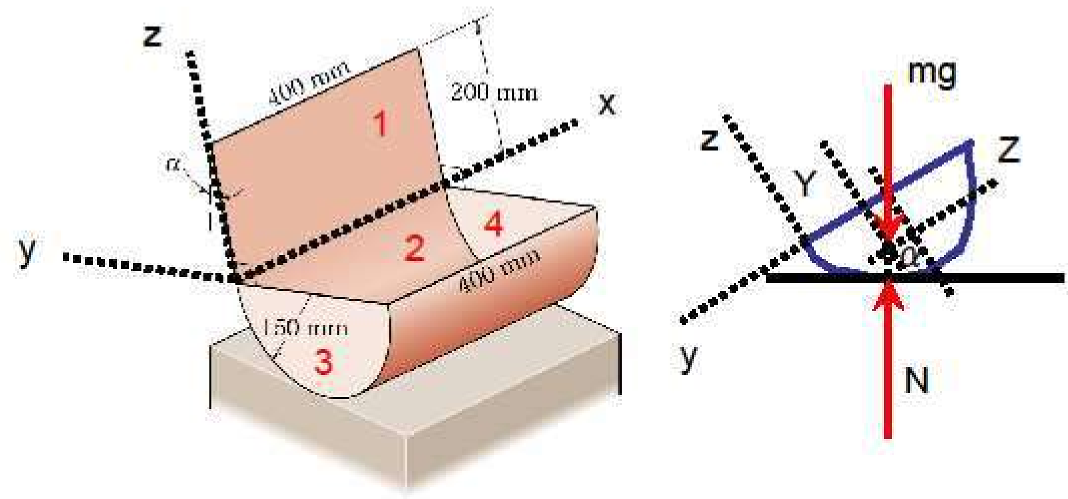


Figure 5.22: Solution to example 5.6

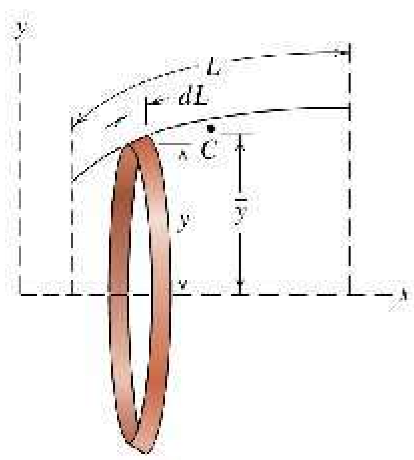


Figure 5.23: Creation of the revolved surface object ([1], pp. 266)

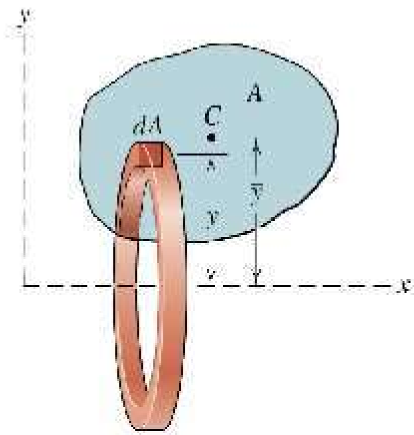


Figure 5.24: Creation of the revolved volume object ([1], pp. 266)

Revolved Surface From fig. 5.23, the infinitesimal area of each strip is

$$dA = \theta y dL$$

using the fact that the surface is constructed by revolving the planar curve L about the x-axis by the angle θ. The infinitesimal area of the strip then would be the length, θy, times the width, dL. Since the other strips are constructed by revolving the segment of the line dL with the same angle θ, the total area of the revolved surface would be

$$A = \theta \int y dL = \theta \overline{y} L \tag{5.8}$$

The rightmost expression is obtained by recalling the line centroidal equation, eq. (5.4). $\overline{y}$ is the y-centroidal coordinate of the curve of length L. Therefore the area may be viewed as the lateral area of a cylinder with length L and radius $\overline{y}$.

Revolved Object From fig. 5.24, the infinitesimal volume of each strip is

$$dV = \theta y dA$$

using the fact that the object is constructed by revolving the planar area A about the x-axis by the angle θ. The infinitesimal volume of the strip then would be the extruded length, θy, times the cross sectional area, dA. Since the other strips are constructed by revolving the differential area dA with the same angle θ, the total volume of the revolved object would be

$$V = \theta \int y dA = \theta \overline{y} A \tag{5.9}$$

The rightmost expression is obtained by recalling the area centroidal equation, eq. (5.5). $\overline{y}$ is the y-centroidal coordinate of the planar area A. Therefore the

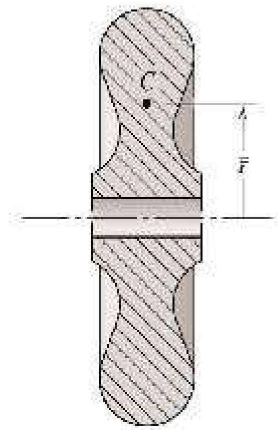


Figure 5.25: Example 5.7 ([2], pp. 280)

volume may be viewed as the extruded volume of the cross sectional area A along the circular path of the centroid (with the length $\theta\overline{y}$) about the revolving axis.

The usage of this theorem may be for directly determining the area or volume of the revolving object. The other usage comes from viewing the equation in the opposite way. Had we know the surface area or volume of the object by some means (possibly from the experiment), we may apply eq. (5.8) or (5.9) to determine the corresponding centroid of the planar curve or the planar area.

Example 5.7 ([2], Prob. 5/89) A hand-operated control wheel made of aluminum has the proportions shown in the cross-sectional view. The area of the total section shown is $15,200$ mm^2, and the wheel has a mass of 10 kg. Calculate the distance $\overline{r}$ to the centroid of the half-section. The aluminum has a density of 2.69 Mg/m^3.

Solution: The mass and density of the control wheel are given. Hence its volume can be determined as

$$[V = M/\rho] \qquad V = 10/2690 = 0.00372 \ \text{ m}^3$$

Since the volume is created from the full-round revolution of half of the section depicted in fig. 5.25, the planar area to revolve is $\frac{15200}{2}\times 1\text{E-6} = 7600\text{E-6} \ \text{ m}^2$. From the Pappus theorem, the distance $\overline{r}$ to the centroid can be calculated as

$$[V = \theta\overline{r}A] \qquad 0.00372 = 2\pi\times\overline{r}\times 7600\text{E-6}$$

$$\overline{r} = 77.85 \ \text{ mm}$$

Example 5.8 ([1], Prob. 5/93) Calculate the mass m of concrete required to construct the arched dam shown. Concrete has a density of 2.40 Mg/m^3.

Solution: The concrete mass can be determined from the knowledge of the density and the volume of the dam. From fig. 5.26, this particular dam is constructed as if we rotate the cross section *A*-*A* about the pole of the 200 m radius circle for 60°. Therefore, we may use Pappus theorem in determining the volume.

First, to avoid direct integration of the differential volume, the centroidal coordinate with respect to the axis of revolution is necessary. To avoid direct integration of the differential area, we view the cross sectional area as obtained by subtracting the quarter area from the square area. Hence the centroidal coordinate can be determined from the known centroids of the composite areas in a discrete manner as

$$\left[\overline{y} = \frac{\sum_i y_{ci} A_i}{\sum_i A_i}\right] \qquad \overline{y} = \frac{80\times80\times(40+120)-\frac{\pi}{4}\times70^2\times\left(\frac{4\times70}{3\pi}+120\right)}{80\times80-\frac{\pi}{4}\times70^2}$$

$$\overline{y} = 175.52 \text{ m}$$

Now the Pappus theorem may be used to calculate the revolved volume as

$$[V = \theta\overline{y}A] \qquad V = \frac{\pi}{3}\times175.52\times\left(80\times80-\frac{\pi}{4}\times70^2\right)$$

$$V = 468,985.2 \text{ m}^3$$

Consequently, the concrete mass required may be determined readily as

$$[m = \rho V] \qquad m = 2400\times468985.2 = 1.126\text{E9 kg}$$

5.6 Fluid Statics

In this section, we study an application of the distributed forces. When an object is surrounded with fluid substance, force will be developed at the object surface due to the pressure of the substance. When the fluid is in static equilibrium, there will be no flow. This requires the shear force to be nulled to satisfy the equilibrium condition. However, there will be compressive force from the fluid acting along the normal direction of the surface. We would like to determine such force.

Consider the free body diagram of an infinitesimal volume of fluid shown in fig. 5.27. Assume the fluid is in equilibrium. With this condition, the fluid pressure at a specified point in any direction will be the same. By the definition of the pressure, which is the ratio of the force to the area acting upon,

$$p = \frac{F}{A} \tag{5.10}$$

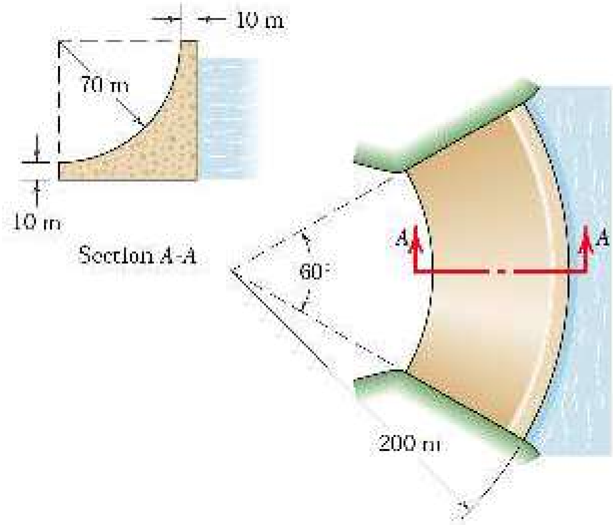


Figure 5.26: Example 5.8 ([1], pp. 273)

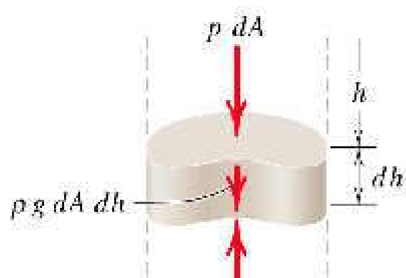


Figure 5.27: Free body diagram of an infinitesimal fluid ([1], pp. 308)

we then can apply it to determine the force acting upon the infinitesimal area dA on the surface of the volume.

Note that all pressure forces in every direction except the vertical one cancels. From the free body diagram and the equilibrium condition, the pressure force on the lower face must balance the summation of the force acting on the upper face and the fluid's weight. This can be reduced to the following relation:

$$dp = \rho g dh \tag{5.11}$$

where ρ is the density of the fluid, g is the gravitational acceleration, and dh is the infinitesimal height measured along the gravitational acceleration direction.

It is observed that the pressure is the function of the dimension in the vertical direction solely. Additionally, the pressure increases with the depth. In fact, with a constant ρ, we may integrate the above relation to determine the pressure as follow:

$$p = p_0 + \rho g h = \text{absolute pressure} \tag{5.12}$$

p_0 is the pressure at $h = 0$. Normally p_0 is selected to be the pressure at the sea water surface level, which is called the *atmospheric pressure*. Mostly, however, we offset p_0 to be 0 (zero). Consequently the pressure p is no longer be the *absolute (or actual) pressure*. Rather, it will be the relative pressure to the atmospheric pressure. This is the same value as that read off the pressure sensor. Therefore this pressure p is commonly called the *gage pressure*.

$$p = \rho g h = \text{gage pressure} \tag{5.13}$$

The unit of pressure is the unit of force over the square of the unit of dimension. For the SI unit system, it turns out to be N/mm^2. To pay tribute to Pascal, a French mathematician who introduce the notion of the pressure, 1 N/mm^2 is equal to 1 Pa.

Now we are in the position ready to determine the *resultant force due to the pressure distribution on the surface*. To simplify the development, we divide the surface into two categories: the flat surface and the curved surface. Furthermore, for the curved surface, we confine the scope to the study of the cylindrical surface with constant width only. Let's begin with the flat surface case.

a) Flat Surface Imagine a flat surface of arbitrary shape immersed in the fluid medium making the angle θ with respect to the vertical downward direction as in fig. 5.28. We would like to determine the resultant force due to the pressure distribution over the whole surface. To achieve this, we could divide the flat surface into horizontal strips along the depth direction. This is because the pressure is constant along the same strip. See fig. 5.28.

The net force acting on a particular strip can be calculated by recalling the pressure definition in eq. 5.10 as

$$dR = pdA = \rho g h(y)\, x(y)\, dy$$

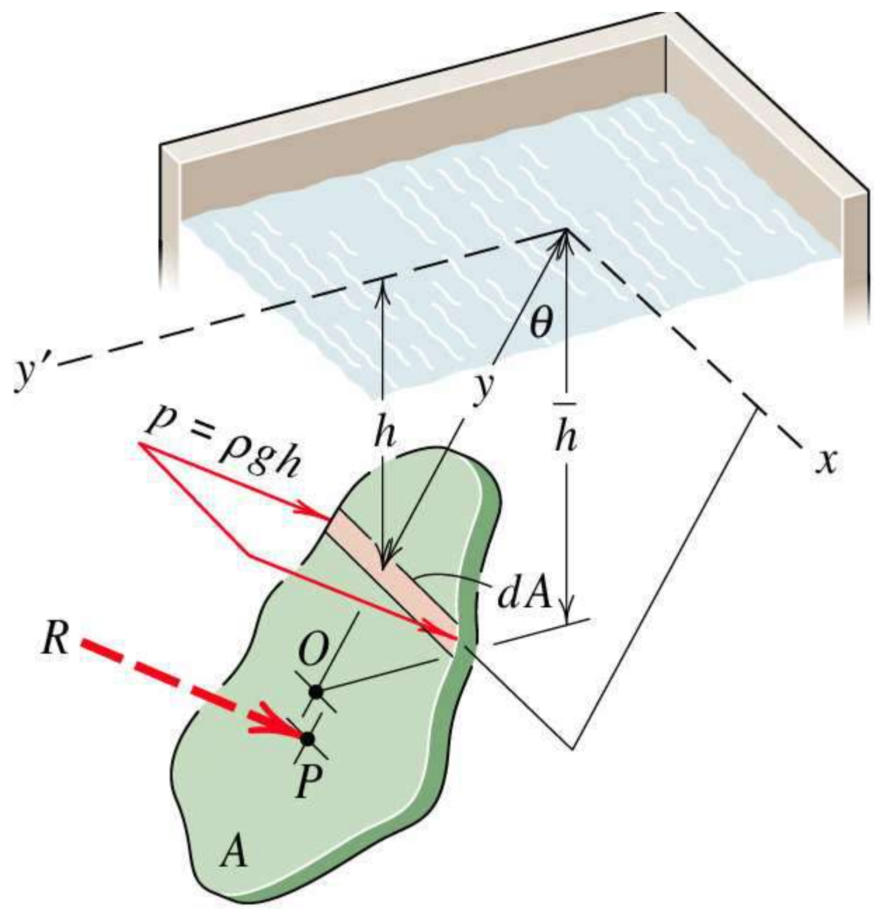


Figure 5.28: A flat surface immersed in the fluid medium with the pertinent coordinates ([1], pp. 312)

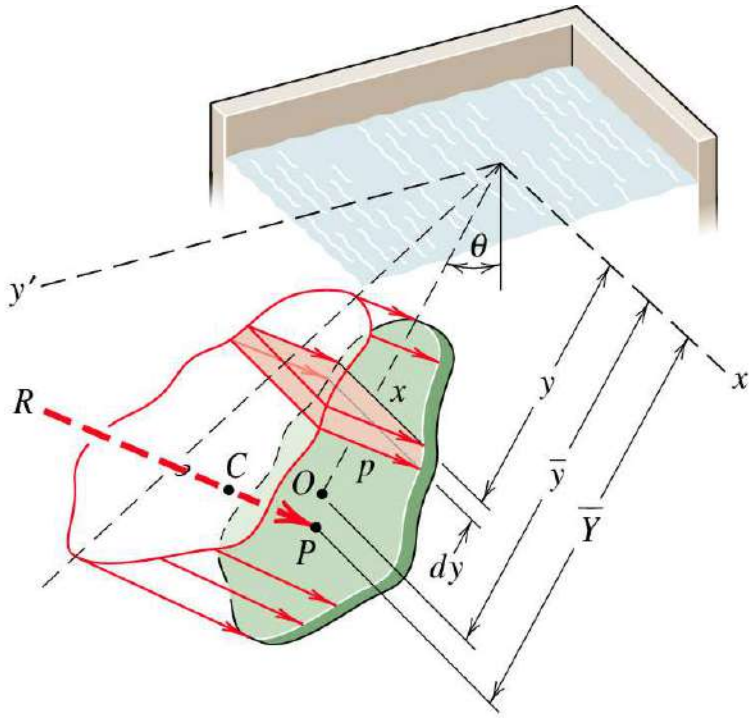


Figure 5.29: Pressure distribution overlaying the flat surface ([1], pp. 312)

where y is the distance measured along the projection of the vertical line onto the surface. x is the width of a specific strip at the y position. Lastly, h is the vertical distance of the strip measured from the surface level of the fluid. Note that the pressure at this strip is $\rho g h\left(y\right)$. Hence the force acting on the strip can be viewed as the volume of the extruded strip with the extruded length be equal to the pressure at that strip.

Since the pressure force acts perpendicular to the surface and the surface is flat, we can obtain the total force over the whole surface by summing the forces acting on all strips. In continuous domain, this will become the integrating operation:

$$R = \int dR = \int \rho g h\left(y\right) x\left(y\right) dy \tag{5.14}$$

The meaning of this formula is significant. The magnitude of the resultant force can be obtained by stacking up the extruded strips over the area of the flat surface and determining the summing volume. If we draw the pressure of each strip on top of it, shown in fig. 5.29, we will have the pressure distribution over the surface. Looking from the side view, the shape is seen to be trapezoidal (fig. 5.30)

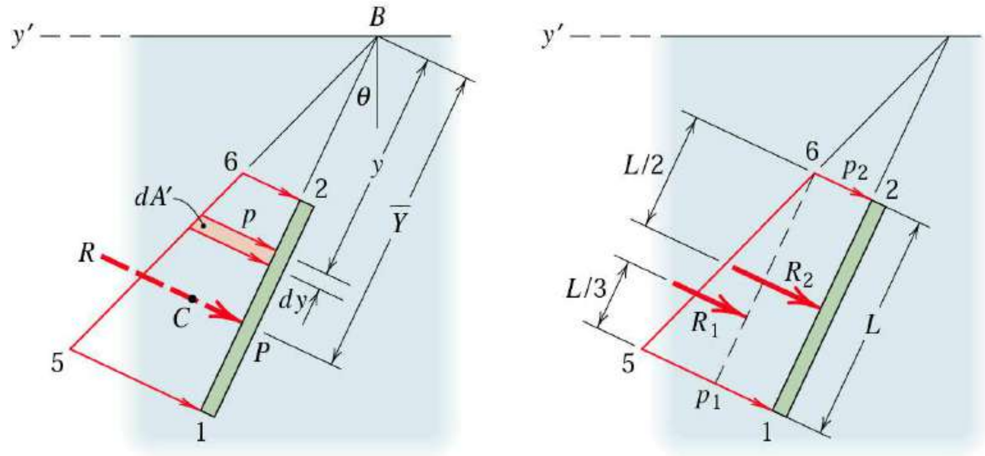


Figure 5.30: Side view profile of the pressure distribution ([1], pp. 310)

due to the linear variation of pressure along the depth, $p = \rho gh$. Therefore we can interpret **the determination of the magnitude of the resultant force to be equivalent to calculating the volume of the pressure distribution**. This fact allows one to avoid evaluating the direct integration had the volume of the pressure distribution been readily recognized. The trapezoidal sectional shape may assist the volume calculation, and furthermore determination of the line of action of the resultant force.

Revisiting the above derivation and making use of the centroidal coordinate definition, we can write

$$R = \rho g \int h dA = \rho g \overline{h} A = p_{av} A \tag{5.15}$$

Hence we have another interpretation of the resultant force. **The magnitude of the resultant force can be thought to be the pressure force induced by the constant pressure, of the value at the centroid of the flat surface area, acting on the whole flat plate**. This allows one to avoid evaluating the integration had the centroid and the area been easily determined.

The direction of the resultant force for the flat surface case is simple because all the forces acting on the infinitesimal strips are parallel to each other and perpendicular to the surface. Consequently, the direction of the resultant force is perpendicular to the surface as well. Nevertheless, little effort is needed for determining the line of action. Applying the principle of moment (Varignon's theorem) and subsituting for the expression of the local force acting on each strip,

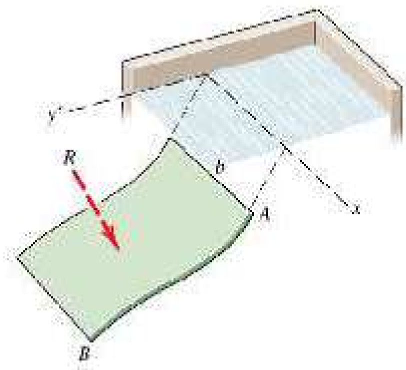


Figure 5.31: A cylindrical surface with constant width immersed in the fluid medium ([1], pp. 311)

$$[R\overline{Y} = \int ydR] \qquad \overline{Y} = \frac{\int ypxdy}{\int pxdy} \tag{5.16}$$

The implication of this equation can be obtained from the principle. **Line of action of the resultant force will pass through the centroid of the volume of pressure distribution** (denoted as point C in fig. 5.29). The above formula yields the y-coordinate of the centroid. Additionally, the piercing point of the line of action to the flat plate is called the *center of pressure* (denoted as point P in fig. 5.29). This point is the projection of the centroid of the volume of pressure distribution along the pressured force direction onto the plate. Note that it is usually not the same point as the centroid of the plate area.

A special case is when the width of the plate is constant, i.e. the rectangular flat plate. In other words, x is constant. This simplifies the above expression for determining the line of action to be

$$\overline{Y} = \frac{\int ydA'}{\int dA'} \tag{5.17}$$

where $dA' = pdy$ is the infinitesimal area of the trapezoid along the y-coordinate. See fig. 5.30. For this case, **line of action of the resultant force will pass through the centroidal y-coordinate of the area A' of the side view profile of the pressure distribution**. The location of this trapezoidal centroid may be evaluated using technique of composite bodies of triangle and rectangle, as depicted in fig. 5.30.

Next let's analyze a specific case of the curved surface, the cylindrical surface with constant width.

b) Cylindrical Surface with Constant Width In this case, the surface is formed as if it is a bended paper sheet. See fig. 5.31 for the cylindrical

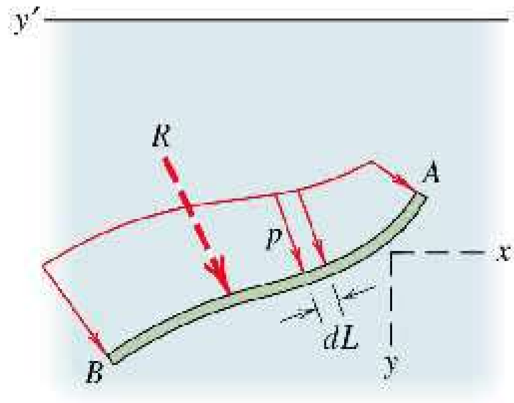


Figure 5.32: Side view profile of the pressure distribution on the cylindrical surface ([1], pp. 311)

surface of the constant width b immersed in the fluid medium. We would like to determine the resultant force $\mathbf{R}$ acting on it.

Similar approach as the flat surface case may be applied here where the plate is first divided along the side view profile into infinitesimal strips (see fig. 5.32). Then the corresponding force magnitude acting on a strip becomes

$$dR = pdA = \rho gh\,(L)\,bdL$$

where L is the curved distance measured along the side view profile of the cylindrical surface. h is the vertical distance of the strip measured from the surface level of the fluid. Again, $\rho gh\,(L)$ is the pressure at the strip of location L. Hence the magnitude of force acting on the strip can be viewed as the volume of the extruded strip with the extruded length be equal to the pressure at that strip.

The pressure forces act perperdicular to the strips. However, they are not parallel as in the case of flat surface which mean that the magnitude of the resultant force cannot be obtained simply by summing the force magnitude of every strip altogether. Instead, we have to project the force acting on each strip into two common components and then sum the components algebraically. The resultant force may in turn be determined from summing vectorially these two components back. If x- and y- components have been chosen as denoted in fig. 5.32, then the magnitude of the resultant force along each direction may be written as

$$R_x = b\int (pdL)_x = b\int pdy \tag{5.18}$$

$$R_y = b\int (pdL)_y = b\int pdx \tag{5.19}$$

Note that we **cannot** evaluate the magnitude of the resultant force by calculating the volume of pressure distribution, or by using eq. 5.15. This is because the pressure forces acting perpendicular to the strips are not parallel to each other and so cannot be summed algebraically. As a result, besides the magnitude, we need to specify its direction and line of action. The direction can be easily determined, for example, by the angle the vector made with the x-direction;

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) \tag{5.20}$$

The line of action may be determined by applying the definition of centroid to evaluate the x- and y-centroidal coordinates separately. From fig. 5.32, the x-centroidal coordinate of R_y is determined from the principal of moment

$$\left[R_y\overline{X} = \int x dR_y\right] \qquad \overline{X} = \frac{\int x dR_y}{\int dR_y} = \frac{\int xpdx}{\int pdx} \tag{5.21}$$

Similarly, the y-centroidal coordinate of R_x may be determined as

$$\left[R_x\overline{Y} = \int y dR_x\right] \qquad \overline{Y} = \frac{\int y dR_x}{\int dR_x} = \frac{\int ypdy}{\int pdy} \tag{5.22}$$

The intersection of the line of action of R_x and R_y will be a point through which the resultant force passes.

An alternative method of determining the resultant force $\mathbf{R}$ on the cylindrical surface, which is easier for most cases, is to apply the **equilibrium condition on block of liquid**. Casting this condition is sensible from the intrinsic assumption of the fluid static, where the liquid is not flowing and incompressible. One can then think casually the block of liquid as a rigid body object.

A typical free body diagram of a block of liquid column over the cylindrical surface viewing from the side is shown in fig. 5.33. In the figure, $\mathbf{P}_x$ and $\mathbf{P}_y$ are the pressure resultant forces acting on the top and side faces of the liquid block. $\mathbf{W}$ is the weight of the liquid block passing through the centroid of the area ABC (the plate has constant width). Note that all of them can be determined readily compared to directly evaluating the resultant force. The reaction force $\mathbf{R}$ from the cylindrical surface acting on liquid block may be determined from the equilibrium conditions. Force equilibrium condition will reveal the magnitude while the moment equilibrium condition will help answer the line of action.

Buoyancy Often, we are interested in the total reaction force acting by the surrounding fluid on the immersed object. Instead of pursuing the previous method of determining the reaction force on the surfaces, a more efficient way exists. Consider fig. 5.34a) where a block of fluid is depicted with the virtual closed boundary surface. This fluid block can maintain its shape because it exerts reaction forces on the surrounding fluid. If this fluid block is taken out, the reaction forces may be seen as shown in fig. 5.34b). This implies there will be the counter-reaction forces acting back from the surrounding fluid onto the

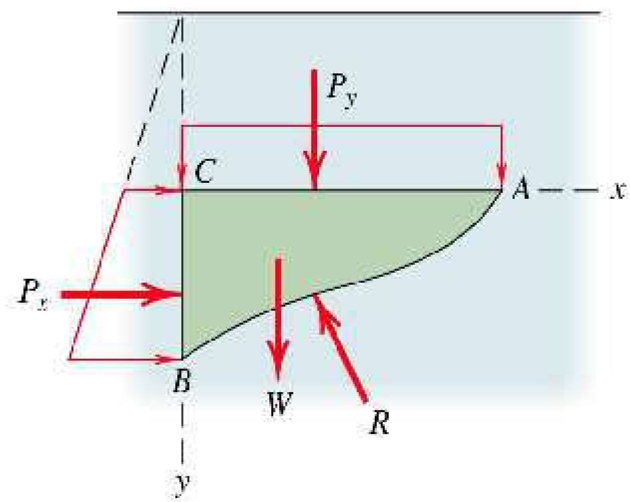


Figure 5.33: Free body diagram of a block of liquid column over the cylindrical surface ([1], pp. 311)

fluid block surface. By the equilibrium condition, this force would have the same magnitude as the weight of the fluid block and act vertically upward through the C.M. of the fluid lump. See fig. 5.34c). Hence, this force is called the *buoyancy force*. **This force is not to be changed for if the fluid block be replaced with any immersed object of the same shape**. Therefore, regarding to the buoyancy force, the following can be concluded

1. The buoyancy force is the pressure resultant forces exerting by the surrounding fluid on the surface of the immersed object.
2. Its magnitude is equal to the weight of the *fluid displaced*.
3. Its line of action passes through the center of mass of the *displaced fluid*.

The magnitude of the buoyancy force may be written as

$$B = \rho_f V_f g \tag{5.23}$$

where ρ_f and V_f are the density and volume of the displaced fluid, respectively. Suppose the material of the object has the density ρ_o, and if the object is fully immersed, its weight will be equal to $W = \rho_o V_f g$. When $\rho_o < \rho_f$, the object's weight will be less than the buoyancy force. This makes the object to buoy up to the surface level and possibly expose to the air, which can be think of as another fluid medium. It will rise up to the equilibrium position where the summation of the buoyancy forces be equal to the object's weight.

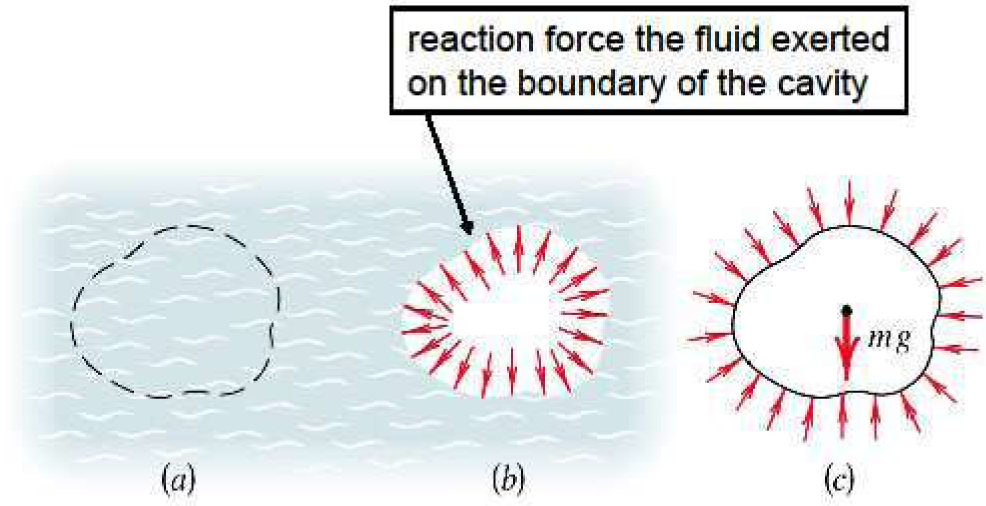


Figure 5.34: Concept of buoyancy force ([1], pp. 314)

Example 5.9 ([1], Prob. 5/179) A deep-submersible diving chamber designed in the form of a spherical shell 1500 mm in diameter is ballasted with lead so that its weight slightly exceeds its bouyancy. Atmospheric pressure is maintained within the sphere during an ocean dive to a depth of 3 km. The thickness of the shell is 25 mm. For this depth calculate the compressive stress σ which acts on a diametral section of the shell, as indicated in the right-hand view.

Solution: The outside sea water induces pressure resultant compressive force on the outer surface of the submarine. In turn, it causes internal stress in the shell as depicted by the free body diagram of the vehicle sectioned in hemisphere in fig. 5.36. Note that only the force component in horizontal direction is essential for calculating the compressive stress. The horizontal compressive force F can be determined by summing the x-component of the pressure force:

$$F = \int (pdA)_x = \rho g \int (hdA)_x = \rho g \overline{h} A_y$$

By equilibrium condition, the pressure force magnitude has to be equal to the compressive force inside the shell material.

$$[\textstyle\sum F_x = 0] \qquad \rho g \overline{h} A_y - \int \sigma dA = 0$$

Substituting the depth of the centroid of the circular area, $\overline{h}$, of which its value is 3000 m into the above equation and assuming the constant compressive

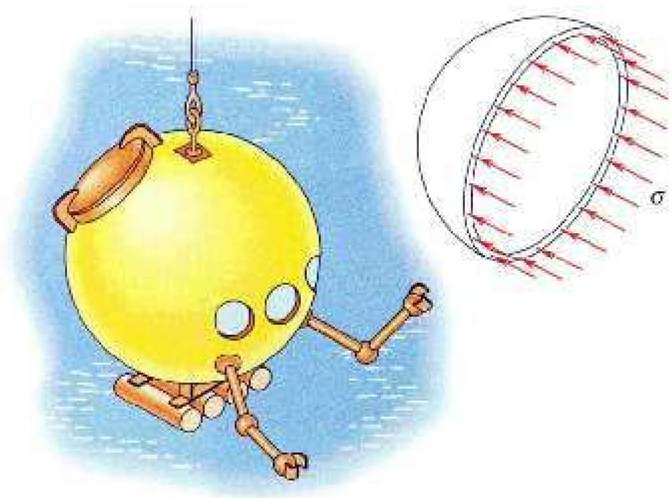


Figure 5.35: Example 5.9 ([1], pp. 320)

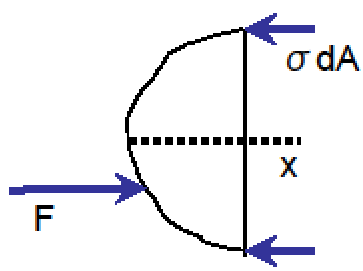


Figure 5.36: Solution to example 5.9

stress, σ, throughout the cross-sectional area, the required stress would be

$$\sigma = \frac{\rho g \overline{h} A_y}{A_{ring}} = \frac{1030 \times 9.81 \times 3000 \times \pi \times 0.75^2}{\pi \times 0.75^2 - \pi \times 0.725^2} = 462.4 \ \text{MPa}$$

Example 5.10 ([2], Prob. 5/169) The cross section of a fresh-water tank with a slanted bottom is shown. A rectangular door 1.6×0.8 m in the bottom of the tank is hinged at A and is open against the pressure of the water by the cable under a tension P as shown. Calculate P.

Solution: Assume that the door has negligible weight and volume. Therefore its weight and buoyancy force may be omitted. To determine P, we need to calculate the pressure force acting on the gate and apply the equilibrium condition to it. Fig. 5.38 shows the free body diagram of the gate and the corresponding pressure distribution and forces acting on it. Since the door

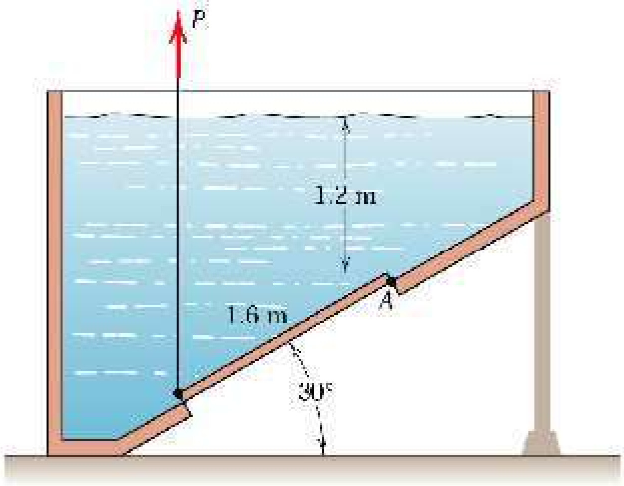


Figure 5.37: Example 5.10 ([2], pp. 328)

surface is flat and has constant width, it is appropriate to calculate the pressure force magnitude from the volume of pressure distribution. Furthermore, we may divide the volume into two parts – rectangular and triangular volume. Hence the indicated force magnitudes are

$$F_1 = (1000 \times 9.81 \times 1.2) \times 1.6 \times 0.8$$
$$F_2 = \frac{1}{2} \times (1000 \times 9.81 \times 1.6 \sin 30^\circ) \times 1.6 \times 0.8$$

Taking the moment of all forces around point A, the balancing tension P may be determined as

$$[\sum M_A = 0] \qquad -P \times 1.6 \cos 30^\circ + F_1 \times 0.8 + F_2 \times 1.6 \times \tfrac{2}{3} = 0$$

$$P = 12566 \text{ N}$$

Example 5.11 ([1]) A rectangular wood block of density 800 kg/m^3 floats stably in the salted water, of which its surface is topped with the spilled oil. The salt water and the oil have density of 1030 and 900 kg/m^3, respectively. Determine the height, h, of the block that is not sunk.

Solution: If all of the pressure forces are referred to the atmospheric pressure forces, the free body diagram of the block may be drawn as shown in fig. 5.40. Recall that the buoyancy force is equal to the weight of the displaced fluid, we separate it into two parts; one from the oil pressure and the other from the salt water pressure. Establishing the equilibrium condition in the vertical

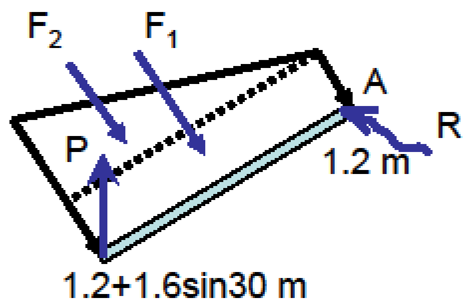


Figure 5.38: Solution to example 5.10

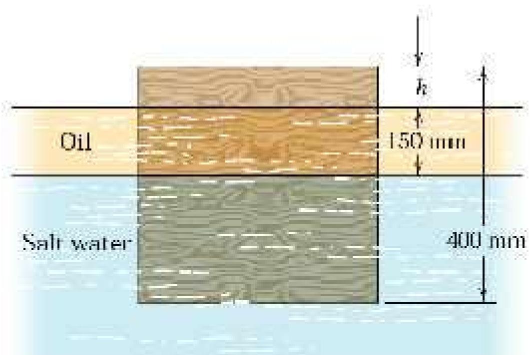


Figure 5.39: Example 5.11 ([1])

direction, the protuding height may be determined.

$$[\sum F_y = 0] \quad 900g \times A \times 150 + 1030g \times A \times (250 - h) - 800g \times A \times 400 = 0$$

$$h = 70.4 \ \ \text{mm}$$

Example 5.12 ([2], Prob. 5/173-174) A channel-marker buoy consists of a 2.4 m hollow steel cylinder 300 mm in diameter with a mass of 90 kg and anchored to the bottom with a cable as shown. If $h = 0.6$ m at high tide, calculate the tension T in the cable. Also find the value of h when the cable goes slack as the tide drops. The density of sea water is 1030 kg/m^3. Assume the buoy is weighted at its base so that it remains vertical.

If the C.M. of the buoy is in the geometric center of the cylinder, calculate the angle α which would be made by the buoy axis with the vertical when the water surface is 1.5 m above the lower end of the cylinder. Neglect the diameter compared with the length when locating the center of buoyancy.

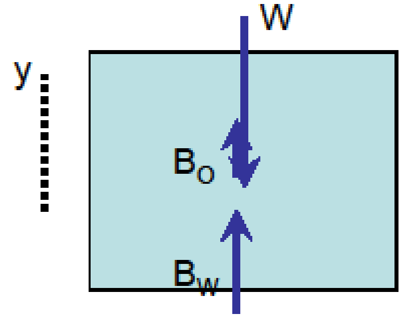


Figure 5.40: Solution to example 5.11

Solution: For the first part, it is assumed that the buoy is weighted at its base, and so it remains vertical. After drawing the free body diagram of the buoy in fig. 5.42a), we may apply the equilibrium force condition along the vertical direction,

$$[\sum F_y = 0] \qquad B - Mg - T = 0$$

For the case when $h = 0.6$ m, the buoyancy force would be determined as

$$B = \rho V g = 1030 \times \frac{\pi \times 0.3^2}{4} \times 1.8 \times 9.81 \quad \text{N}$$

Substituting the values into the above equilibrium equation, we can solve for the cable tension

$$T = 402.7 \quad \text{N}$$

When the cable goes slack, $T = 0$ and the buoyancy force must decrease to counter the buoy's weight solely. Consequently, the new height h must increase. The new equilibrium equation would become

$$[\sum F_y = 0] \qquad B - Mg = 0$$

$$1030 \times \frac{\pi \times 0.3^2}{4} \times (2.4 - h) \times g - 90 \times g = 0$$

$$h = 1.164 \quad \text{m}$$

In the second part, the buoy's weight passes through its centroid, showing in the new free body diagram of fig. 5.42b). Therefore, the buoy might be tilted instead of just upright only. Let the tilted angle be θ. From the free body diagram, the weight acts through the buoy's centroid and the buoyancy force passes through the centroid of the immersed portion. We then can set up the equilibrium in moment about the base point of the buoy as follow

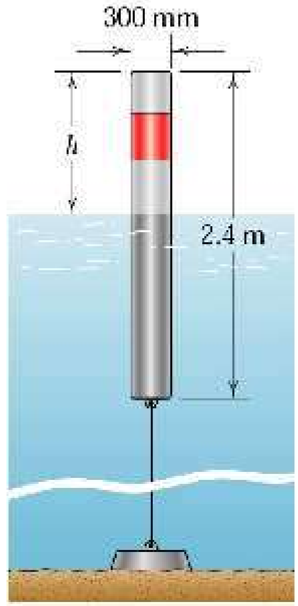


Figure 5.41: Example 5.12 ([2], pp. 329)

$[\sum M_o = 0]$ $$B \times 0.75 \tan\theta - 90 \times g \times 1.2 \sin\theta = 0$$

The buoyancy force magnitude is determined from the weight of the displaced fluid:

$$B = \rho V g = 1030 \times \frac{\pi \times 0.3^2}{4} \times \frac{1.5}{\cos\theta} \times g$$

Substitute this value into the above equilibrium condition, it requires

$$\sin\theta = 0 \quad \text{or} \quad \cos^2\theta = 0.7584$$

Therefore,

$$\theta = 0^\circ \quad \text{or} \quad \pm 29.44^\circ$$

Example 5.13 ([1], Prob. 5/186) A fresh water channel 3 m wide is blocked at its end by a rectangular barrier, shown in section ABD. Supporting struts BC are spaced every 0.6 m along the 3 m width. Determine the compression C in each strut. Neglect the weights of the members.

Solution: The free body diagram of the barrier is shown in fig. 5.44 showing the pressure distribution, the corresponding resultant force, the compressive strut forces and the supporting force at the hinge. The magnitude of the

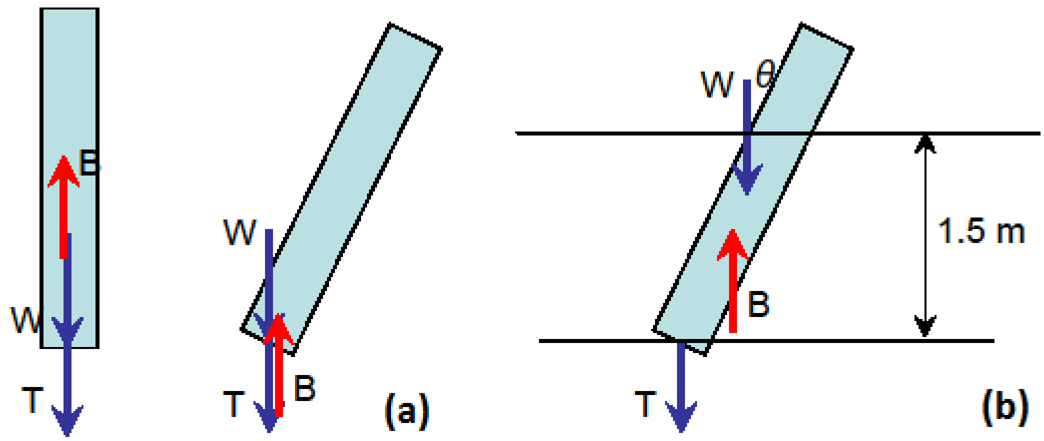


Figure 5.42: Solution to example 5.12

resultant force may be determined from the volume of the pressure distribution which is

$$R = 0.5 \times 1.2 \times (1000 \times 9.81 \times 1.2 \sin 60^\circ) \times 3 = 18.351 \quad \text{kN}$$

Substituting this value into the moment equilibrium condition about the hinge point,

$$[\textstyle\sum M_A = 0] \qquad -R \times 0.4 + 6C \sin 60^\circ \times 0.6 = 0$$

$$C = 2.354 \quad \text{kN}$$

Example 5.14 ([1], Prob. 5/193) The barge crane of rectangular proportions has a 6×15 m cross section over its entire length of 40 m. If the maximum permissible submergence and list in sea water are represented by the position shown, determine the corresponding maximum safe mass m_o that the barge can handle at 10 m extended position of the boom. Also find the total displacement m in the metric tons of the unloaded barge. The distribution of machinery and ballast places the CG of the barges, minus the mass m_o, at the center of the hull.

Solution: Free body diagram of the crane reveals the pertinent forces and make clear the equilibrium posture of it. The buoyancy force B is equal to the weight of the displaced fluid,

$$B = \rho V g = 1030 \times 15 \times 6 \times \frac{40}{2} \times g$$

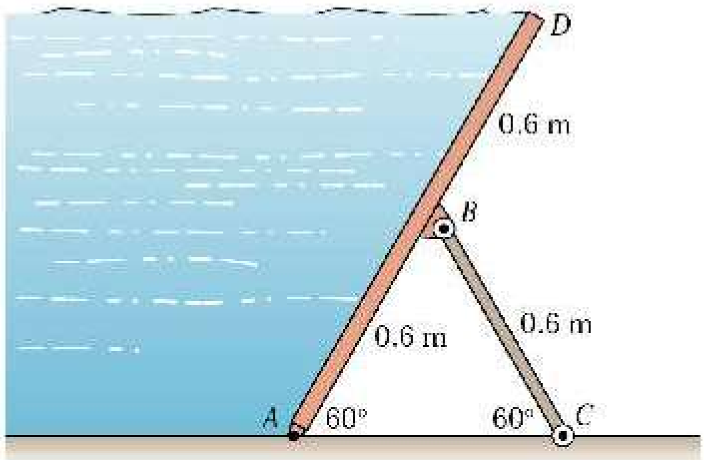


Figure 5.43: Example 5.13 ([1], pp. 322)

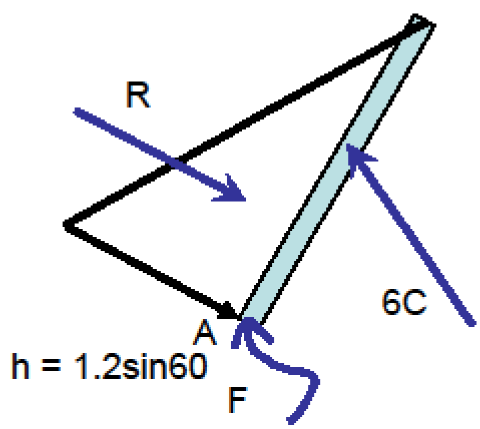


Figure 5.44: Solution to example 5.13

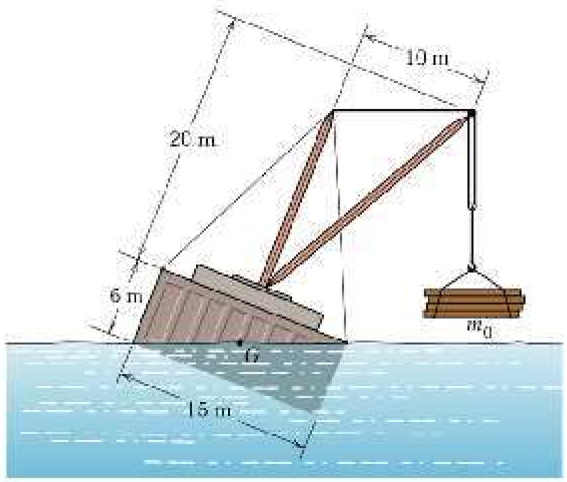


Figure 5.45: Example 5.14 ([1], pp. 324)

and acts through its C.M. The submerged portion is of prismatic shape and hence the easiness of determining the location of its centroid, as indicated by the intersection of the 2 m horizontal line and the 5 m vertical line shown in fig. 5.46. The maximum permissible safe mass may be determined from setting the moment equilibrium condition of the crane critical posture about the C.G. of the barges.

$$[\sum M_G = 0] \qquad B\cos\theta \times 2.5 - B\sin\theta \times 1 - m_o g \times (10\cos\theta + 23\sin\theta) = 0$$

$$m_o = 203 \ \text{ Mg}$$

The unloaded weight of the barge itself may be determined using the force equilibrium condition along the vertical direction;

$$[\sum F_y = 0] \qquad B - mg - m_o g = 0$$

$$m = 1651 \ \text{ Mg}$$

Example 5.15 ([1], Prob. 5/197) The fresh water side of a concrete dam has the shape of a vertical parabola with vertex at A. Determine the position b of the base point B through which acts the resultant force of the water against the dam face C.

Solution: This problem falls to the case of cylindrical surface with constant width. Therefore, we will use the alternative method of free body diagram of the block of fluid since it is more convenient. The free body diagram of the

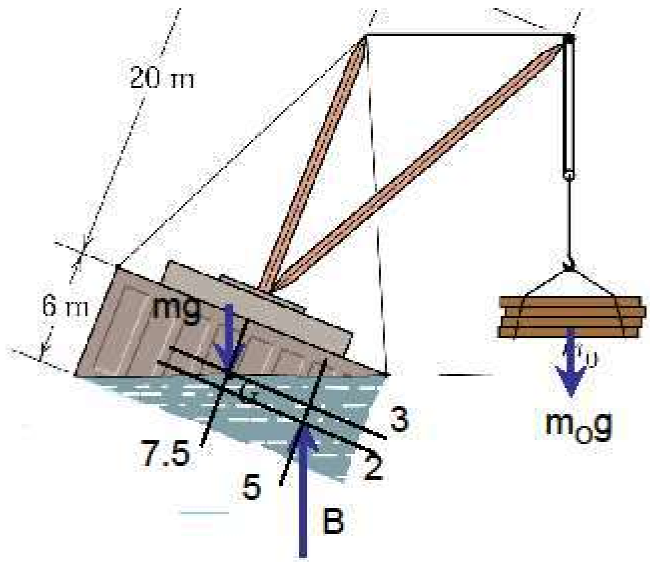


Figure 5.46: Solution to example 5.14

fluid column on the dam face C is shown in fig. 5.48. First, we will determine the fluid weight. The shape of the dam surface can be described by a vertical parabola with the vertex at A. If we set the x-y coordinate frame with the origin at A, the surface may be written in equation of the form $y = ax^2$ or $x = \sqrt{y/a}$ in the first quadrant.

From fig. 5.47, the dam surface passes through $(27, 36)$. Substitute this point into the equation, we can solve for the coefficient

$$a = \frac{y}{x^2} = \frac{36}{27^2}$$

To determine the fluid weight, we use the density definition which requires the knowledge of the fluid volume. The fluid volume may be calculated from the cros-sectional area times the depth of the fluid block. See fig. 5.48.

$$[V = Ah] \qquad V = \left(\int_0^{36} x dy\right) h = \left(\tfrac{2}{3}xy|_{y=0}^{y=36}\right) h = 648h \ \ \mathrm{m}^2$$

Therefore the fluid weight W is

$$[W = \rho V g] \qquad W = 1000 \times 9.81 \times 648h$$

Its line of action passes through the centroid of the volume. This point projected onto the cross-sectional view will be coincident with the centroid of the area. Hence the distance $\overline{X}$ shown may be determined as

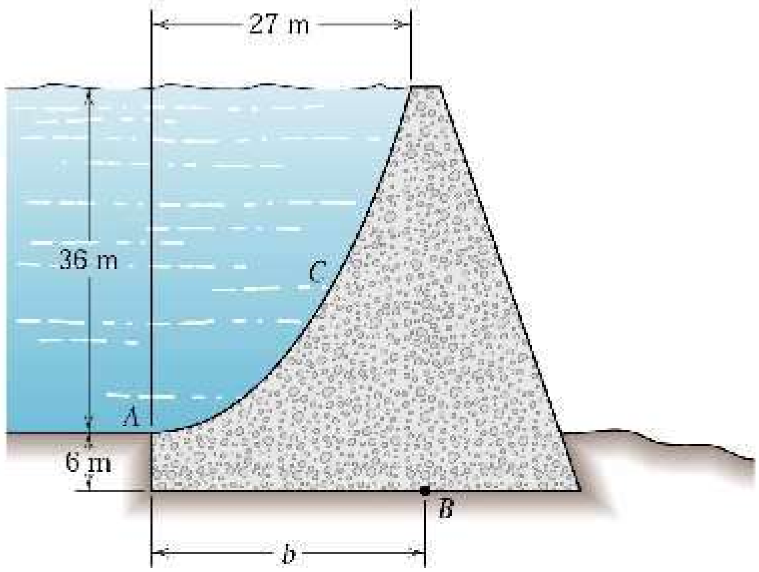


Figure 5.47: Example 5.15 ([1], pp. 325)

$[A\overline{X} = \int x_c dA]$ $$648\,\overline{X} = \int_0^{36} \left(\frac{x}{2}\right) x dy$$

$$\overline{X} = 10.125 \quad \text{m}$$

There is the pressure force from the sided fluid acting on the vertical left surface. We may determine its magnitude from the volume of pressure distribution,

$$F = \frac{1}{2} \times 36 \times (1000 \times 9.81 \times 36) \times h$$

The line of action passes through the centroid of the volume of pressure distribution which, for this simple geometry, is 12 m above the vertex point A.

We decompose the resultant force from the dam acting on the parabolic surface passing through B into the horizontal and vertical components. It is obvious that the magnitude of R_y must be equal to the fluid weight. From the figure, R_x passes through point D which is located at the distance b to the left of point B. Moment of all forces about D must be balanced. This condition is used to determine the distance b.

$[\sum M_D = 0]$ $$-F \times 18 - W \times 10.125 + W \times b = 0$$

$$b = 28.125 \quad \text{m}$$

Example 5.16 ([1], Prob. 5/220) A flat plate seals a triangular opening in the vertical wall of a tank of liquid of density ρ. The plate is hinged about the upper

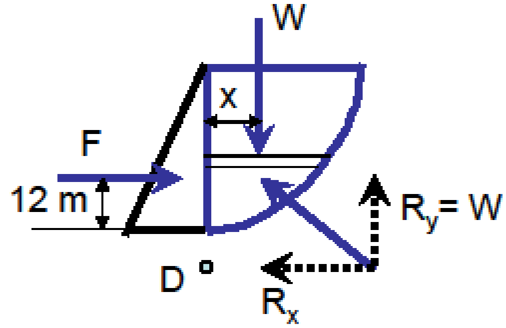


Figure 5.48: Solution to example 5.15

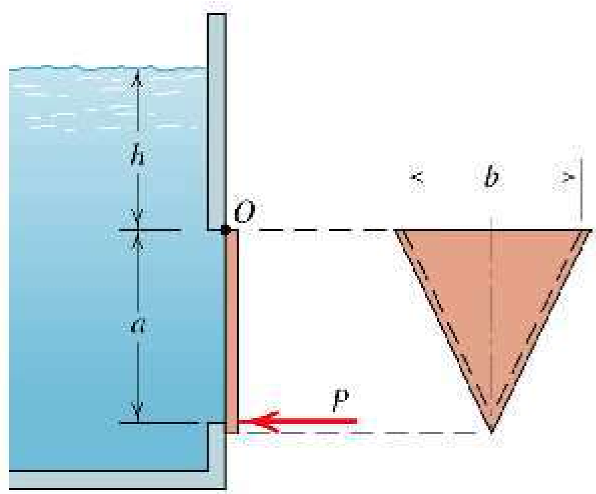


Figure 5.49: Example 5.16 ([1], pp. 333)

edge O of the triangle. Determine the force P required to hold the gate in a closed position against the pressure of the liquid.

Solution: This flat plate has varying width along the depth. Hence the explicit formula of the volume of pressure distribution is not available. One might resort to determining the volume from integrating the volume of the infinitesimal strip over the surface and formulating the moment equilibrium condition about O. However, an extra step of evaluating the line of action of the pressure force must be performed.

Rather, since the problem does not ask for the pressure force. We may determine the holding force P by formulating the moment equilibrium condition about O and dealing with the infinitesimal distributed force directly. See fig. 5.50. A particular infinitesimal force dR acting at the depth of y below the hinge edge

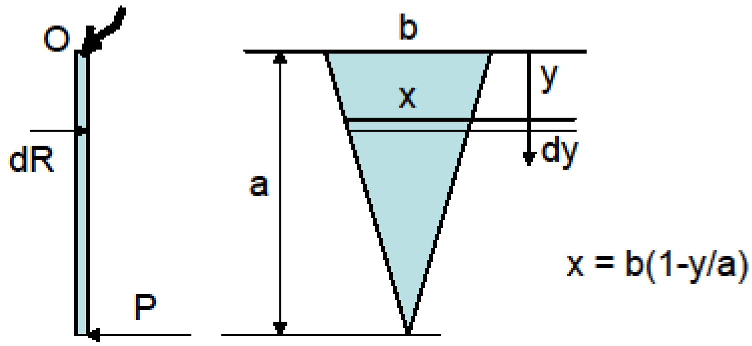


Figure 5.50: Solution to example 5.16

may be expressed as

$$dR = pdA = \rho g \left(h + y\right) xdy$$

The width x must be expressed in terms of the changing variable y. From the geometric relationship of similar triangle,

$$\frac{x}{a-y} = \frac{b}{a} \quad \rightarrow \quad x = b\left(1 - \frac{y}{a}\right)$$

This leads to

$$dR = \rho g \left(h + y\right) \times b\left(1 - \frac{y}{a}\right) dy$$

These infinitesimal forces induce the moments around O, for which their summation must balance the moment produced by the holding force P. Therefore

$$[\sum M_o = 0] \qquad -Pa + \int_{y=0}^{y=a} ydR = 0$$

Substitute the expression for dR into the equation and evaluate the integral,

$$P = \frac{\rho gba}{6}\left(h + \frac{a}{2}\right)$$

Chapter 6

Introduction to Dynamics

6.1 Basic Concepts

In this section, we roughly explain the basic technical terms that will be used throughout the course.

- **Space** is the region occupied by the bodies. We set up an *coordinate system* to specify where the object is by the *position* and its posture by the *orientation*. Measurement of the absolute values must be done relative to the *inertial* (fixed) reference frame. For the practical engineering problem, where the magnitude of the velocity is small compared to that of the earth, motion of the object can be calculated using the earth's frame as the fixed reference frame with negligible error. And we can assume those measurement *absolute*.
- **Time** is the measure of the succession of events. Often, we are more interested in the change of physical quantities with respect to time, e.g. $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, instead of time variable itself.
- **Mass** is the measure of the *inertia* of a body. The inertia indicates the resistance to a change in motion.
- **Force** —a fixed vector— is the measure of the attempt to move a body.
- **Particle** is a body of which its dimension is negligible. The rotation effect is insignificant because it is just a point. Whether the body can be treated as the particle or not depends on the relative dimensions in the problem and how much detailed of the solution we are interested in.
- **Rigid body** is a body whose relative movement between its parts are negligible relative to the gross motion of the body. For example the motion of an ingot can be analyzed by assuming the object being rigid.
- **Nonrigid body** is a body whose relative movement between its parts are significant relative to the gross motion of the body. Knowledge of the mechanics of the deformable material must be used along with Dynamics in order to determine the absolute motion of the nonrigid bodies.

Let us consider some examples to see the difference of each term. If we have an object and consider the very small substance of the body. For differential element analysis of the body, the small substance can be treated as a particle. However, the substance must be handled as connecting objects had the molecular effects in the body are of concern. Or think of an airplane. Even of its huge size, the whole airplane may be modeled as a point in flight speed analysis along the route. But if the rotational motion, such as yawing or pitching, of the airplane body is important, its size does matter.

The next two examples are to show whether an object is considered rigid or nonrigid depends on how much detailed of the problem we would like to analyze. Truss can just be looked as a rigid body for the preliminary design of truss structure. Buf we must think of the truss elasticity if we were to choose the material for that truss. A stiff linkage of the robot may be considered a rigid body. However, the n-connecting linkages, treated as a whole, to form the robot arm is an example of nonrigid body. Note the body-fixed inertia of the nonrigid body is not constant.

6.2 Newton's Laws

In this section, we briefly mention the Newton's laws that describe the motion of the particle under low velocity. The **first law** states:

> "A *particle* remains at rest or continue to move in a straight line with a uniform velocity if there is no unbalanced force acting on it."

This statement can be formulated as

$$\Sigma \mathbf{F} = \mathbf{0} \Leftrightarrow \mathbf{a} = \mathbf{0}$$

Newton's **second law**, the most well-known of three, states:

> "The *absolute* acceleration of a *particle* is proportional to the resultant force acting on it and is in the direction of this resultant force."

This statement can be formulated as

$$\Sigma \mathbf{F} = m\mathbf{a} \tag{6.1}$$

where $\mathbf{a}$ = *absolute* acceleration of the particle.

Newton's **third law** states:

> "The forces of *action* and *reaction* between interacting bodies are equal in magnitude, opposite in direction, and collinear."

It can be nmemonically written as

$$\text{action force} = -\,(\text{reaction force})$$

This fact is used very often in drawing the free body diagram (FBD).

6.3 Gravitational Law

Any two bodies have the attraction force governed by the gravitational law

$$F = G\frac{m_1 m_2}{r^2} \tag{6.2}$$

where

F = attraction force
G = gravitational constant value = $6.673 \times 10^{-11}\ \ \mathrm{m^3/\left(kg \cdot s^2\right)}$
m = mass of the involving bodies
r = distance between the bodies

Hence there is always the attraction force between the earth and the object. This gravitational force is called the *weight* of the body.

$$W = m\frac{Gm_e}{r^2} = mg \tag{6.3}$$

where

g = free falling acceleration observed on the **moving** earth
= $9.81\ \ \mathrm{m/s^2}$

In practice, however, the gravitational acceleration can be considered the **absolute** acceleration for the engineering problem on earth.

Chapter 7

Kinematics of Particles

7.1 Introduction

Kinematics is the study of the motion of bodies with no consideration to the forces that accompany the motion. Kinematics analysis is a prerequisite to kinetics, which is the study of the relationship between the motion and the corresponding forces that causes the motion or are generated as a result of the motion.

In this chapter, we are interested in kinematics of particles. A particle is a body whose physical dimensions are so small compared with the radius of curvature of its path. Therefore an airplane may be considered as a particle if the problem of interested is related to the flying route between two cities, even the size of the airplane is huge compared to a human. In other words, the body rotation effect is insignificant compared to the translation.

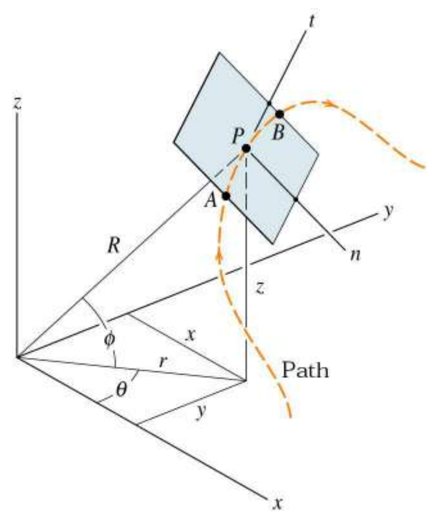


Figure 7.1: Several descriptions of the particle position ([3], pp. 22)

The implication of treating the object as a particle is that the concept of rotation does not exist, in contrast to the time it is treated as a body. This simplifies the analysis to a great degree. In the following, let us consider a particle P moving along a specified trajectory depicted in fig. 7.1. Fundamentals of kinematics is the position of the particle.

Position of P The problem is to describe where the particle currently is. There are several ways to achieve this. One might use the distance along any three mutually perpendicular directions, denoted as x, y, and z; that is we may use the standard *rectangular coordinate system*.

Another way is to project the particle onto a certain plane for which the origin and a reference direction are predetermined. The distance from the origin to the projected point is denoted as r. The angle measured from the reference to the radial direction is called θ. And the distance measured along the line perpendicular to the plane from the projected point to the particle is called z. $\{r, \theta, z\}$ forms the *cylindrical coordinate system* that might be used to describe the position.

Yet another way to specify where the particle be is to employ the *spherical coordinate system*. This is possible by first determining the origin, the reference plane, and the reference direction. The distance measured from the origin to the point P is called R. Next we project this distance vector onto the reference plane. The angle measured from the reference line to the projected radial vector is noted as θ, or the azimuth angle. Finally, we measure another angle from the radial vector to the distance vector. This angle is called the elevation angle, denoted as ϕ.

Each description has its own advantage and hence is suitable to some particular tasks. For example, the spherical coordinate system is largely employed in navigation system, such as rocket, radar, or satellite system. The cylindrical coordinate system might be appropriate for some robots that have this structure for pick and place operation. The rectangular coordinate system is suitable for general tasks since it is easiest to understand the position of an object in three dimensions by telling the width, height, and depth.

Motion of P We are not satisfied only just being able to describe where the particle is because it is moving. We must be able to describe the motion of the particle. In other words, we will also be interested in the change of its position. This is achieved in general by differentiating the position with respect to time. Before the analysis, first we must decide which kind of the coordinate system we will be using. Other than intrinsically different types of coordinate systems described above, they may also be categorized according to whether the frames used are fixed (not moving) or moving.

<u>Fixed reference coordinate frame</u> In this case, the coordinates are measured with respect to the fixed reference frame. For example, we are standing on the ground and seeing the airplane flying. That is we are observing the motion of the airplane from the fixed reference frame. What we are measuring is really the absolute motion of the particle. Hence this is called *absolute motion* analysis.

Unfortunately, often the motion is convolved. It is quite inconvenient to describe the position of the particle using the fixed reference frame solely. In such situation, it is more advantage to employ the moving reference coordinate frame.

<u>Moving reference coordinate frame</u> Here the coordinates will be measured with respect to the moving reference frame. For an appropriate reference frame, the description of the complex motion may be simplified. For example, we may

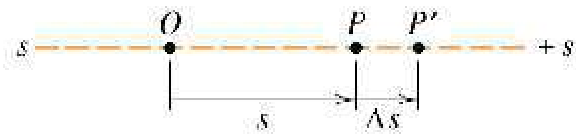


Figure 7.2: Rectilinear motion showing the distance s and its change Δs measured along the straight line path ([3], pp. 22)

fly on another plane following the target plane in a manner that the relative location between them is fixed. Consequently, we will see the plane not moving in this moving reference frame! What we are measuring is just the motion of the particle relative to the moving frame. When we add this *relative motion* to the motion of the reference frame, we will recover the absolute motion.

7.2 Rectilinear Motion

To embark on the general three dimensional motion of a particle immediately might be too demanding for the beginners. Therefore we shall start from the simplest kind of motion, i.e. rectilinear motion. Rectilinear motion is the motion along a straight line. The particle is constrained to move along a particular straight line. In this simple case, it is unnecessary to employ full version of coordinate systems since the only independent variable that will be changed during the course is the coordinate measured along the line. This coordinate will then be selected as the *position coordinate* for rectilinear motion, denoted by the symbol s shown in fig. 7.2.

Let the change in the position coordinate occurred during the time interval Δt be the *displacement* Δs. Note that both the position coordinate and the displacement can be negative value had they go opposite to the predefined positive direction. The *average velocity*, v_{av}, is defined to be

$$v_{\text{av}} = \Delta s/\Delta t \tag{7.1}$$

The *velocity* or, to be precise, the instantaneous velocity is the average velocity as the time interval Δt approaches zero. Recalling the definition of the differentiation, this may be written as

$$v = \lim_{\Delta t \to 0} \Delta s/\Delta t = \frac{ds}{dt} = \dot{s} \tag{7.2}$$

In words, the velocity is the time rate of change of the position coordinate.

We can do what we just did with the position coordinate to the velocity to obtain the acceleration. Let the change in the velocity occurred during the time interval Δt be Δv. The *average acceleration*, a_{av}, is defined to be

$$a_{\text{av}} = \Delta v / \Delta t \tag{7.3}$$

The *acceleration* or, to be precise, the instantaneous acceleration is the average acceleration as the time interval Δt approaches zero. Recalling the definition of the differentiation, this may be written as

$$a = \lim_{\Delta t \to 0} \Delta v / \Delta t = \frac{dv}{dt} = \dot{v} = \frac{d^2 s}{dt^2} = \ddot{s} \tag{7.4}$$

In words, the acceleration is the time rate of change of the velocity. Multiplying eq. 7.2 and 7.4 together, and integrating with respect to time, we obtain the relationship:

$$v dv = a ds \tag{7.5}$$

which is sometimes useful as we shall see.

Some facts about position, velocity, and acceleration

1. Displacement is a vector quantity. In fact, it is a free vector directing from the initial position and ending at the final position.
2. On the contrary, distance is a positive scalar quantity. It is measured along the (possibly curved) path the distance that the particle really has travelled.
3. Both velocity and acceleration are free vectors. Therefore their changes happen from two sources; change in magnitude and change in direction.

In solving the problems, one may need to perform the integration to the definitions and the relationship of kinematics parameters in eq. 7.2, 7.4, and 7.5. Some of their physical meanings are, for example,

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v dt \quad \rightarrow \quad s_2 - s_1 = \text{area under } v\text{−}t \text{ curve}$$

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a dt \quad \rightarrow \quad v_2 - v_1 = \text{area under } a\text{−}t \text{ curve}$$

$$\int_{v_1}^{v_2} v dv = \int_{s_1}^{s_2} a ds \quad \rightarrow \quad \frac{v_2^2 - v_1^2}{2} = \text{area under } a\text{−}s \text{ curve}$$

$$v = \frac{ds}{dt} \quad \rightarrow \quad v = \text{slope of } s\text{−}t \text{ curve}$$

$$a = \frac{dv}{dt} \quad \rightarrow \quad a = \text{slope of } v\text{−}t \text{ curve}$$

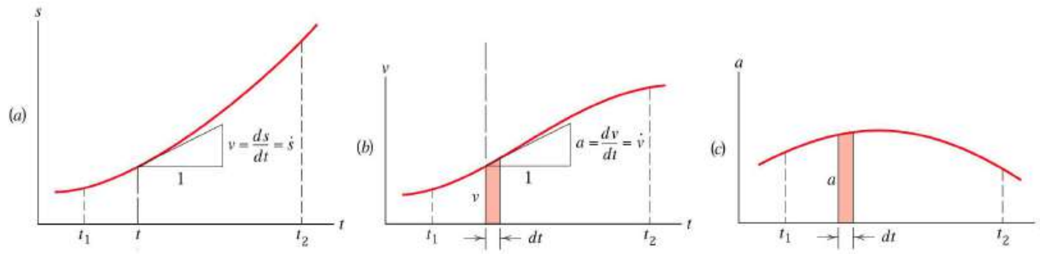


Figure 7.3: Plots of the displacement, velocity, and acceleration with respect to time and their related quantities ([3], pp. 24)

These relationships are depicted in fig. 7.3. The interpretation suggests an alternative way of numerical methods in evaluating the kinematic parameters.

Typically the acceleration (in general could be a function of displacement, velocity, or time explicitly) of the object will be given. Then the problem will ask for other kinematical parameters. Conceptually, these can be determined by integrating the acceleration and solve for the quantities, such as velocity, position, or time. To make the reader become acquainted with such problems, in the following, we will consider several case studies where the given acceleration be functions of other kinematic quantities.

a) $\underline{a = \text{constant}}$ Common forces that lead to the constant acceleration are the gravitational force, or dry friction force, for example. Without loss of generality, we assume at the beginning the following quantities: $t = 0$, $s = s_o$, and $v = v_o$. Then at a later time t, we would have

$$\int_{v_o}^{v} dv = a \int_{0}^{t} dt \quad \rightarrow \quad v = v_o + at$$

Or if the displacement information is known, we may write

$$\int_{v_o}^{v} v dv = a \int_{s_o}^{s} ds \quad \rightarrow \quad v^2 = v_o^2 + 2a\left(s - s_o\right)$$

And the elapsed displacement may be determined from

$$\int_{s_o}^{s} ds = \int_{0}^{t} v dt = \int_{0}^{t} \left(v_o + at\right) dt \quad \rightarrow \quad s = s_o + v_o t + \frac{1}{2} a t^2$$

b) $\underline{a = f\left(t\right)}$ In this case, the acceleration would be the result of the synthetic forces applied for specific purposes. For example, the explosion force in the piston cylinder from appropriate control of the valve timing and fuel amount causes the automobile to run at a desired motion. Compared to the constant acceleration, we *cannot* pull out the acceleration of the integral sign. The pertinent kinematic quantities may then be determined as follow:

$$\int_{v_o}^{v} dv = \int_0^t f(t)\,dt \quad \rightarrow \quad v = v_o + \int_0^t f(t)\,dt$$

$$\int_{s_o}^{s} ds = \int_0^t v dt \quad \rightarrow \quad s = s_o + \int_0^t v dt = s_o + v_o t + \int_0^t \int_0^\theta f(\zeta)\,d\zeta d\theta$$

by substituting the velocity expression as a function of time into subsequent integration. Alternatively, one may see the problem

$$\ddot{s} = f(t) \text{ , with i.c. } t_o = 0,\ s_o,\ v_o$$

as a second-order ordinary differential equation, and use one of the available techniques in solving the problem directly.

c) $\underline{a = f(v)}$ Some apparatus provides the force that is explicitly a function of the component's velocity, such as the viscous drag force in the dashpot. Therefore the corresponding acceleration will be related to the velocity. The velocity and displacement may then be determined by

$\left[a = \frac{dv}{dt}\right]$ $\qquad t = \int_0^t dt = \int_{v_o}^{v} \frac{1}{f(v)} dv$

By inversing the relationship, we may write the velocity as a function of time,

$$v = g(t)$$

Integrating the equation with respect to time, the displacement is obtained:

$$s = h(t)$$

On the other hand, if the information on time is not provided, we may derive the displacement relationship from

$[vdv = ads]$ $\qquad \int_{v_o}^{v} \frac{v}{f(v)} dv = \int_{s_o}^{s} ds$

for which we may solve for the displacement as a function of the velocity

$$s = s_o + \int_{v_o}^{v} \frac{v}{f(v)} dv = g(v)$$

d) $\underline{a = f(s)}$ Many devices possess the spring behavior to some degrees. That is, they provide the resistive forces that are function of the deformation. Another type of forces in this category is the field force, such as the attraction force. From the gravitational law, the attraction force will be inversely proportional to the square of the separated distance. These forces induce the acceleration that is a function of the displacement. Hence we may solve for the velocity

$$\int_{v_o}^{v} v dv = \int_{s_o}^{s} f(s)\, ds \quad \rightarrow \quad v^2 = v_o^2 + 2\int_{s_o}^{s} f(s)\, ds$$

as a function of displacement

$$v = g(s)$$

The displacement may be determined explicitly by

$\left[v = \frac{ds}{dt}\right]$ $$t = \int_{s_o}^{s} \frac{1}{g(s)} ds$$

Inverting the above relationship, we may have

$$s = h(t)$$

Following are sample problems related to the rectilinear motion where we need to understand specific hidden implications and apply previous definitions in determining the answers.

Example 7.1 ([3], Prob. 2/23) Small steel balls fall from rest through the opening at A at the steady rate of 2 per second. Find the vertical separation h of two consecutive balls when the lower one has dropped 3 meters. Neglect air resistance.

Solution: The accceleration for the free-falling object must be equal to the constant of gravity. Also, we can integrate for the velocity and the displacement explicitly. In other words,

$$a = g$$

$$v = v_o + gt$$

$$s = s_o + v_o t + gt^2/2$$

Since the ball is dropped from rest at the reference level of the opening, we have the initial conditions

$$v_o = 0, \;\; s_o = 0$$

Hence

$$s = gt^2/2$$

Apply this relation to the ball already dropped by 3 m, the time spent would be

$$3 = gt_l^2/2, \;\; t_l = 0.782 \;\; \text{s}$$

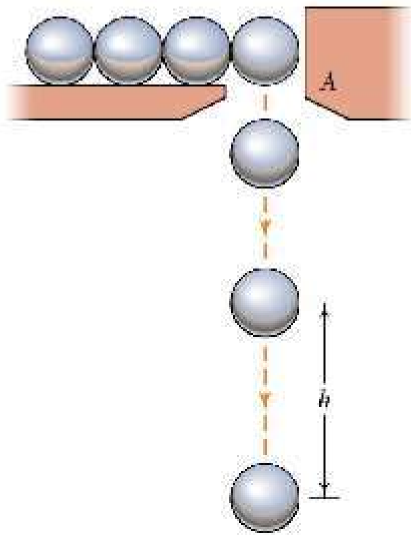


Figure 7.4: Example 7.1 ([3], pp. 33)

Since the balls are released with the rate of 2 per second, the time the consecutive ball spent so far would then be

$$t_u = t_l - 0.5 = 0.282 \quad \text{s}$$

Use the above relation again to solve for the distance travelled, which is related to the vertical separation h:

$$s_u = 3 - h = gt_u^2/2 \quad \rightarrow \quad h = 2.61 \quad \text{m}$$

as depicted in fig. 7.5.

Example 7.2 ([4], Prob. 2/23) In traveling a distance of 3 km between points A and D, a car is driven at 100 km/h from A to B for t seconds and at 60 km/h from C to D also for t seconds. If the brakes are applied for 4 s between B and C to give the car a uniform deceleration, calculate t and the distance s between A and B.

Solution: A helpful acceleration and velocity history diagram should be drawn as shown in fig. 7.7. The distance traveled during the interval AB and CD may then be obtained by

$$\left[v = \frac{ds}{dt}\right] \qquad \int_{s_A}^{s_B} ds = \int_0^{t/3600} v dt = \int_0^{t/3600} 100 dt, \quad s_B = t/36$$

and

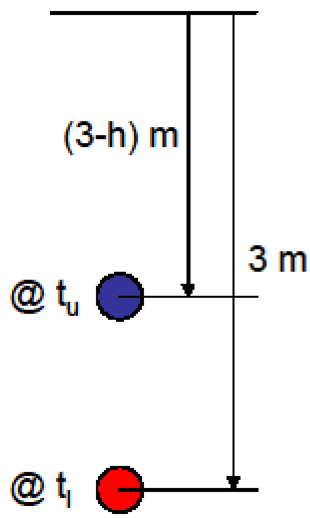


Figure 7.5: Solution to example 7.1

$$\int_{s_D}^{s_C} ds = \int_0^{t/3600} v dt = \int_0^{t/3600} 60 dt, \quad 3 - s_C = t/60$$

The distance traveled during the braking may be calculated indirectly from the area under v-t curve as (fig. 7.7)

$$s_C - s_B = \frac{1}{2} \times \frac{4}{3600} \times (100 + 60) = 4/45$$

These three equations may be used to solve for the time spent

$$t = 65.5 \quad \text{sec}$$

Consequently, the distance s between A and B may be determined by substituting the time back into the first relation

$$s = s_B = 1.819 \quad \text{km}$$

Example 7.3 ([3], Prob. 2/27) The 350-mm spring is compressed to a 200-mm length, where it is released from rest and accelerates the sliding block A. The acceleration has an initial value of 130 m/s^2 and then decreases linearly with the x-movement of the block, reaching zero when the spring regains its original 350-mm length. Calculate the time t for the block to go a) 75 mm and b) 150 mm.

Solution: We may draw the change in the acceleration with respect to the

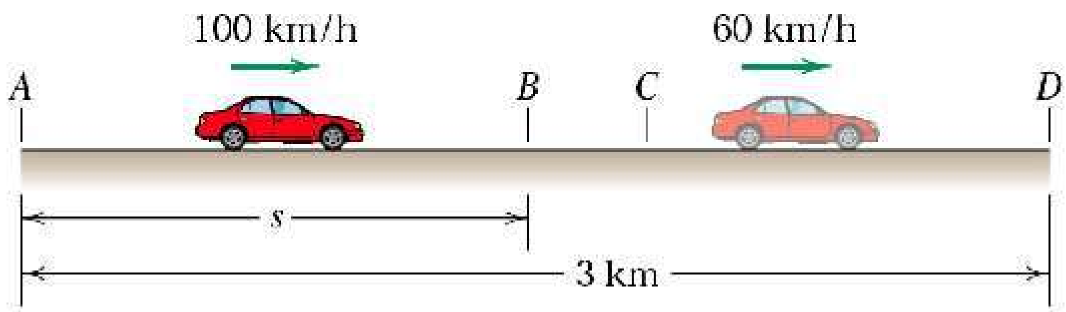


Figure 7.6: Example 7.2 ([4], pp. 30)

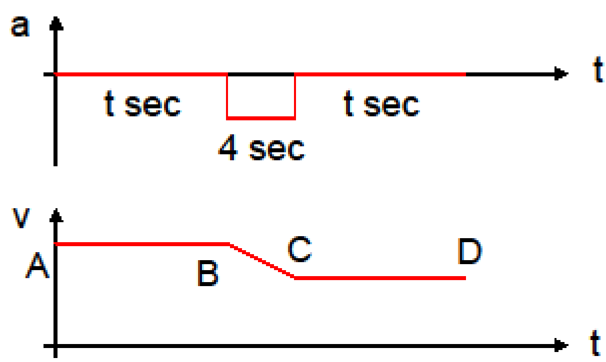


Figure 7.7: Solution to example 7.2

block motion from the system description as shown in fig. 7.9. Accordingly, the acceleration-displacement relationship may be written as

$$a = -\frac{130}{0.15}x = -866.7x$$

This graph inspires us to apply the velocity-displacement relationship through the acceleration:

$$[vdv = ads] \qquad \int_0^v vdv = \int_{-0.15}^x ads = \int_{-0.15}^x -866.7sds$$

$$v^2 = -866.7x^2 + 19.5, \quad v = 29.44\sqrt{0.0225 - x^2}$$

assuming that the block move positively to the right. The time spent for the block to travel to the distance x would then be

$$\left[v = \tfrac{ds}{dt}\right] \qquad \int_0^t dt = \int_{-0.15}^x \tfrac{1}{v}ds = \int_{-0.15}^x \tfrac{1}{29.44\sqrt{0.0225-s^2}}ds$$

$$t = 0.034\left[\sin^{-1}\left(\frac{x}{0.15}\right) + \frac{\pi}{2}\right]$$

By substituting the position of the box $x = 75$ mm, the time spent would be 71.14 ms. For the position at $x = 150$ mm, the time spent is 106.7 ms.

Alternatively, one may use the definition of the acceleration

$$[a = \ddot{s}] \qquad \ddot{x} + 866.7x = 0$$

to formulate the equation which is recognized as the unforced harmonic equation. It has a well-known solution form of

$$x = A\sin\omega t + B\cos\omega t, \quad \omega = \sqrt{866.7} = 29.44 \ \text{ rad/s}$$

We may deduce the initial conditions from the problem statements as follow.

$$x_o = -0.15 \ \text{ m}, \quad \dot{x}_o = 0 \ \text{ m/s}$$

Substituting these conditions into the above solution form, we may be able to solve for the coefficients:

$$A = 0, \quad B = -0.15$$

Therefore, the motion of this block behaves according to

$$x = -0.15\cos 29.44t$$

for which we may inversely solve for the time spent. The answers agree with the other approach.

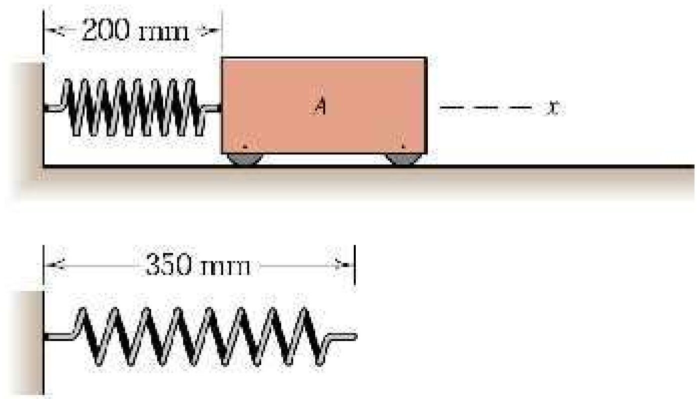


Figure 7.8: Example 7.3 ([3], pp. 34)

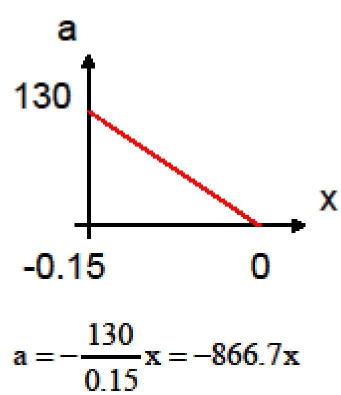


Figure 7.9: Solution to example 7.3

Example 7.4 ([4], Prob. 2/31) A train that is traveling at 130 km/h applies its brakes as it reaches point A and slows down with a constant deceleration. Its decreased velocity is observed to be 96 km/h as it passes a point 0.8 km beyond A. A car moving at 80 km/h passes point B at the same instant that the train reaches point A. In an unwise effort to beat the train to the crossing, the driver 'steps on the gas'. Calculate the constant acceleration a that the car must have in order to beat the train to the crossing by 4 s and find the velocity v of the car as it reaches the crossing.

Solution: Since all objects in the problem move with constant acceleration, we may refer to the instant formula applied to the constant acceleration case, namely,

$$v = v_o + at$$

$$v^2 = v_o^2 + 2a\,(s - s_o)$$

$$s = s_o + v_o t + at^2/2$$

For the train, the driver applies the brake causing the constant deceleration. This makes the velocity being reduced from 130 to 96 km/h for the distance traveled 0.8 km. Therefore the deceleration may be determined:

$$96^2 = 130^2 + 2a \times 0.8, \quad a = -4802.5 \;\; \text{km/h}^2$$

By this deceleration, the train would reach the intersection at the time counted from the time it reaches point A

$$1.6 = 130t - 4802.5t^2/2, \quad t = 0.0189 \text{ hour or } 68.11 \text{ s}$$

For the car to beat the train to the crossing by 4 s, the time it must spent counting from point B must be

$$t = 68.11 - 4 = 64.11 \text{ s or } 0.0178 \text{ hour}$$

To achieve this, the car must be accelerated by the constant value of

$$2 = 80 \times 0.0178 + a \times 0.0178^2/2, \quad a = 3628.3 \;\; \text{km/h}^2 = 0.28 \;\; \text{m/s}^2$$

From this action, the car would have the velocity of

$$v = 80 + 3628.3 \times 0.0178 = 144.6 \;\; \text{km/h} = 40.2 \;\; \text{m/s}$$

when it reaches the crossing.

Example 7.5 ([3], Prob. 2/44) The horizontal motion of the plunger and shaft is arrested by the resistance of the attached disk that moves through the oil bath. If the velocity of the plunger is v_o in the position A where $x = 0$ and $t = 0$,

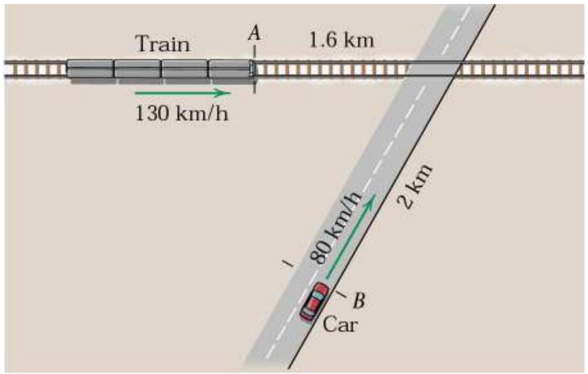


Figure 7.10: Example 7.4 ([4], pp. 32)

and if the deceleration is proportional to v so that $a = -kv$, derive expressions for the velocity v and position coordinate x in terms of the time t. Also express v in terms of x.

Solution: Since the acceleration is given as a function of the velocity, we apply the definition of the acceleration and the velocity directly to determine the velocity and position expressions in terms of the time.

$\left[a = \frac{dv}{dt}\right]$ $$\int_{v_o}^{v} \frac{1}{-kv} dv = \int_0^t dt, \quad v = v_o e^{-kt}$$

$\left[v = \frac{ds}{dt}\right]$ $$\int_0^x ds = \int_0^t v_o e^{-kt} dt, \quad x = \frac{v_o}{k}\left(1 - e^{-kt}\right)$$

To express the velocity in terms of the displacement, we may use the following differential relationship:

$[vdv = ads]$ $$\int_{v_o}^{v} \frac{1}{-kv} v dv = \int_0^x ds, \quad v = v_o - kx$$

Example 7.6 ([4], Prob. 2/44) The electronic throttle control of a model train is programmed so that the train speed varies with position as shown in the plot. Determine the time t required for the train to complete one lap.

Solution: From the velocity-displacement graph, the train travels with the constant velocity of 0.25 m/s in the straight line segments. Hence, the time used for each segment may be simply calculated by

$[\Delta s = vt]$ $$t = \Delta s / v = 2/0.25 = 8 \text{ sec}$$

Next, for a quarter of the circle, i.e. from 2 to $2 + \frac{\pi}{2}$ m, the velocity decreases

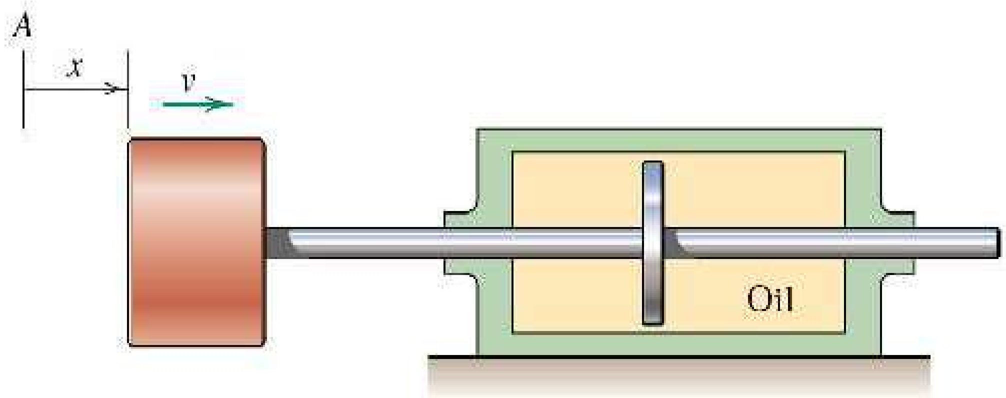


Figure 7.11: Example 7.5 ([3], pp. 37)

uniformly with the traveled distance in a manner that the slope

$$\frac{dv}{ds} = -\frac{0.125}{\pi/2} = -0.25/\pi$$

as seen from fig. 7.12. This slope appear in the reformulation of the relationships

$\left[vdv = ads,\ a = \frac{dv}{dt}\right]$ $\qquad v\frac{dv}{ds} = \frac{dv}{dt}$

which allows us to solve for the elapsed time as

$$\int_0^{\Delta t} (-0.25/\pi)\, dt = \int_{0.25}^{0.125} \frac{1}{v} dv, \quad \Delta t = 8.71 \text{ sec}$$

Motion in other segments replicate the above explained segments. Therefore, total lap time used for one round travel may be determined.

$$\text{lap time} = 8 \times 2 + 8.71 \times 4 = 50.84 \text{ sec}$$

Example 7.7 ([4], Prob. 2/52) A bumper, consisting of a nest of three springs, is used to arrest the horizontal motion of a large mass that is traveling at 40 m/s as it contacts the bumper. The two outer springs cause a deceleration proportional to the spring deformation. The center spring increases the deceleration rate when the compression exceeds 0.5 m as shown on the graph. Determine the maximum compression x of the outer spring.

Solution: The associated deceleration of the mass with respect to the displacement is given. This leads us to apply the following relationship

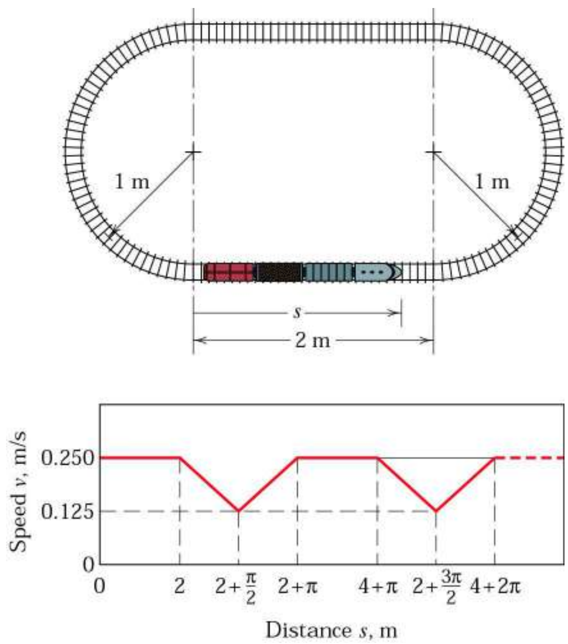


Figure 7.12: Example 7.6 ([4], pp. 35)

$[vdv = ads]$ $\int_{40}^{0} vdv =$ area under $a-s$ curve

In particular,

$$\frac{0-40^2}{2} = -\frac{1}{2} \times 0.5 \times 1000 - \frac{1}{2} \times (x-0.5) \times (1000 + 1000 + 4000\,(x-0.5))$$

for which we recognize the negative area due to the deceleration. Also, we employ the equation of the second line segment in determining the height of the end point. Solving the equation for the maximum compression x corresponding to null motion, we have

$$x = 0.831 \quad \text{m}$$

7.3 Plane Curvilinear Motion

We turn our interest to analyzing the motion of the particle that is more involved. Specifically, the particle will now be moving along an arbitrary curved path that lies in a fixed plane. This is called planar curvilinear motion. Main source of the difficulty roots to the fact that direction of the motion is allowed to change. Consequently, the curvilinear motion must be described by the *vector* quantities that have both the *magnitude* and the *direction* characteristics.

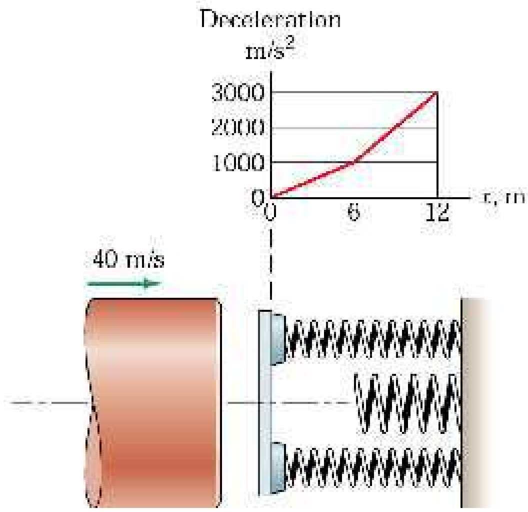


Figure 7.13: Example 7.7 ([4], pp. 36)

An important fact about the vector is that its representation depends on the coordinate system employed. For many cases there are several choices of coordinate systems which may be used. However, only a few are appropriate to the situation. Nevertheless, vector itself is an intrinsic quantity. In other words, it is invariant to the change in coordinate system. Only the representation that does change. In the following sections, we will study some of the common representations.

Before that, we will take a look at the fundamentals of kinematics quantites (i.e. position, velocity, and acceleration) again, except that they will be treated as vector quantities for this time. Note that intrinsically they are coordinate-system-free quantities.

Position Consider a particle A moving along the curvilinear planar path depicted in fig. 7.14. Let there be a point O referring to the fixed reference location. The location of the particle may then be described by a *position vector* $\overline{r}(t)$ (fixed vector type) starting from the reference location and ending at the place where the particle is currently located. At the time t, the particle is at A and the corresponding position vector is $\overline{r}(t)$. After the time Δt has elapsed, the particle has traveled along the path so that currently is at A'. Consequently, the position vector now becomes $\overline{r}(t+\Delta t)$. The *displacement vector* is defined to be the vector difference of the position vectors,

$$\Delta\overline{r} = \overline{r}(t+\Delta t) - \overline{r}(t) \tag{7.6}$$

On the contrary, the *distance* Δs traveled by the particle during this same interval is a positive scalar quantity for measuring the cumulative length along

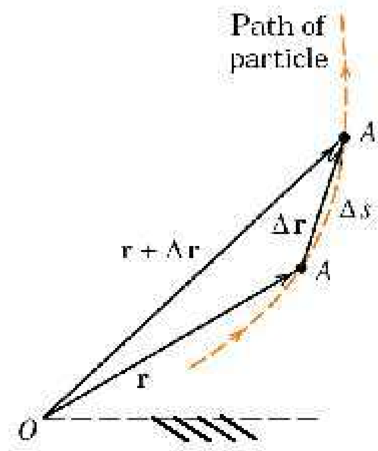


Figure 7.14: Position vector describing the location of the particle ([3], pp. 40)

the curved path. Hence it is always greater than or equal to the magnitude of the displacement vector.

Velocity An indication of how the position vector changed is related to what is called the velocity. Referring to fig. 7.14, motion of the particle has made the change in the position vector $\Delta\overline{r}$ during the interval Δt. Average of this change over time is defined as the *average velocity*

$$\overline{v}_{\text{av}} = \frac{\Delta\overline{r}}{\Delta t} \tag{7.7}$$

Similarly, the average of the distance over time is defined as the *average speed*

$$v_{\text{av}} = \frac{\Delta s}{\Delta t} \tag{7.8}$$

If we shrink the time interval Δt during the analysis in the way that it converges to zero, i.e. the motion occurs in an infinitesimal time interval, the average velocity will be called the *instantaneous velocity* or, more commonly, the *velocity*:

$$\overline{v} = \lim_{\Delta t \to 0} \frac{\Delta\overline{r}}{\Delta t} = \frac{d\overline{r}}{dt} = \dot{\overline{r}} \tag{7.9}$$

by recalling the definition of the differentiation. In other words, the velocity is the average velocity at that instant. Since $\overline{v}$ is derived from the change in the position vector $\overline{r}$, it includes the effect of change both in the magnitude and direction of $\overline{r}$.

Considering fig. 7.14, as $\Delta t \to 0$, the direction of the displacement vector $\Delta\overline{r}$ approaches that of the tangent line to the trajectory. This implies $\overline{v}$ is always a

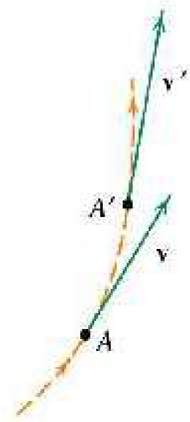


Figure 7.15: Velocity vector always tangent to the path of the particle ([3], pp. 40)

vector that has the direction tangent to the path. Had we know the trajectory of the particle, we will know the direction of the velocity at every instant as drawn in fig. 7.15.

If we consider the magnitude of the displacement alone and treating the time interval $\Delta t \to 0$, the magnitude value of the average velocity would approach

$$|\overline{v}| = \lim_{\Delta t \to 0} \frac{|\Delta \overline{r}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

using the fact that $|\Delta \overline{r}|$ will approach Δs as $\Delta t \to 0$. This leads us to the definition of the *speed*,

$$v = \frac{ds}{dt} = \dot{s} = |\overline{v}| \tag{7.10}$$

Consequently, the speed may be determined from the magnitude of the velocity.

Acceleration Analogous derivation may be performed on the velocity, rather than the position vector, to obtain the acceleration. At any two instant of the motion separated by the time interval Δt, the associated velocity may have changed by the amount $\Delta \overline{v}$. Figure 7.16 depicts the velocity vector diagram. The *average acceleration* is defined to be the change in the velocity over the time interval, namely,

$$\overline{a}_{\text{av}} = \frac{\Delta \overline{v}}{\Delta t} \tag{7.11}$$

As the time interval approaches zero, the average acceleration may be determined by differentiating the velocity. Its value is called the *instantaneous*

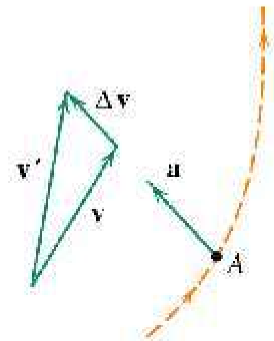


Figure 7.16: Acceleration vector always pointing toward the region containing the center of curvature ([3], pp. 40)

acceleration, or, more commonly, the *acceleration*:

$$\overline{a} = \lim_{\Delta t \to 0} \frac{\Delta \overline{v}}{\Delta t} = \frac{d\overline{v}}{dt} = \dot{\overline{v}} = \ddot{\overline{r}} \tag{7.12}$$

Since $\overline{a}$ is derived from the change in the velocity vector $\overline{v}$, it includes the effect of change both in the magnitude and direction of $\overline{v}$. Additionally, the nice property of the velocity vector that is tangent to the trajectory *does not* propagate to the case of the acceleration. Because the magnitude of the velocity at any point can be arbitrary, generally the direction of the acceleration is neither tangent nor normal to the path. However, it will be pointing toward the half-plane, divided by the tangent line, that contains the (instantaneous) center of curvature of the path. This is due to the fact that the normal component of the acceleration always points toward the center of curvature. More details can be found in section 7.5.

In the following three sections, we will study these three kinematical quantities (position vector, velocity, and acceleration) under three commonly used coordinate systems in plane motion: the rectangular, the normal and tangential, and the polar coordinate systems. Basic kinematical definitions will be employed in deriving the quantities in terms of particular coordinate system configurations and their time rate of changes. As a general guideline for selecting an appropriate coordinate system representation, it should be chosen according to the manner in which the motion is generated or by the form in which the data are specified.

7.4 Rectangular Coordinates $(x - y)$

Rectangular coordinate system is the most commonly used one because its coordinate parameters, denoted x and y, are in accordance with the daily measurement along the wide and long directions. For this coordinate system, we have to set up the coordinate frame which consists of the origin O, and two mutually perpendicular directions denoted by x and y symbols for x- and y-directions. Associated with these directions are the unit vectors $\mathbf{i}$ and $\mathbf{j}$, respectively. From the elementary properties of the vectors, we may write the position vector describing the location of the particle specifically as

$$\overline{r} = x\left(t\right)\mathbf{i} + y\left(t\right)\mathbf{j} \tag{7.13}$$

That is the position vector is decomposed into two subvectors along the x- and y-directions. If these directions are unchanged, we may determine the velocity from its definition:

$$\overline{v} = \dot{\overline{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = v_x\mathbf{i} + v_y\mathbf{j} \tag{7.14}$$

with $\frac{d\mathbf{i}}{dt} = \overline{0}$ and $\frac{d\mathbf{j}}{dt} = \overline{0}$. The acceleration may be determined by differentiating the velocity, of which its straightforward relationship is

$$\overline{a} = \dot{\overline{v}} = \ddot{\overline{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} = a_x\mathbf{i} + a_y\mathbf{j} \tag{7.15}$$

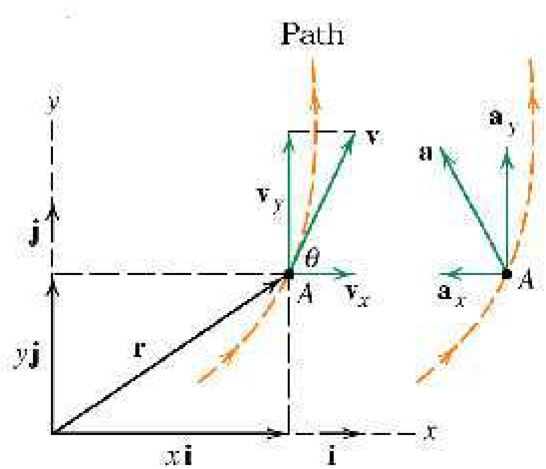


Figure 7.17: Position, velocity, and acceleration described in x-y coordinate system ([3], pp. 43)

From the above kinematic equations, if the x- and y-components of the acceleration are independently generated or determined, i.e. a_x and a_y at a specific point are related by the same instant of time only, we may integrate each individual expression with respect to time to obtain the velocity and position components. One may imagine the curvilinear motion be generated from the superposition of two perpendicular rectilinear motions simultaneously. In fact this kind of motion is found in machines like x-y plotter or the gantry crane in the factory. Therefore the rectangular coordinate system is suitable for the complex motion which is natively generated from the mutually orthogonal rectilinear independent motions simultaneously.

If one has solved for the parametric equations in time $x = f_1(t)$ and $y = f_2(t)$, the equation of the curved path, written in general form as $y = f(x)$, may be determined by algebraically eliminating the time parameter t in those two parametric equations. Consulting the diagram shown in fig. 7.17, some of the common quantities may be calculated from the kinematical vector components as shown below.

$$\text{direction of the velocity}: \quad \tan\theta = \frac{v_y}{v_x} = \frac{dy}{dx} \tag{7.16}$$

$$\text{speed}: \quad v = \sqrt{v_x^2 + v_y^2} \tag{7.17}$$

$$\text{magnitude of the acceleration}: \quad a = \sqrt{a_x^2 + a_y^2} \tag{7.18}$$

Projectile Motion A typical motion that is suitably represented using the rectangular coordinate system is the projectile motion. It is the motion of the thrown object. For the first-run analysis, we shall neglect the aerodynamic drag force. Also it is assumed that the effects of the curvature of the rounded earth and its rotation are small enough. The range of the altitude of the whole motion is in the order that the gravitational force variation may be omitted.

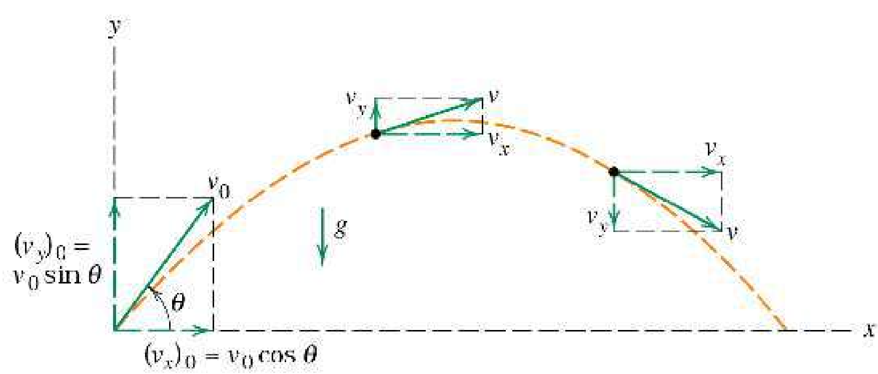


Figure 7.18: Motion profile of the projectile motion showing variations in the velocity ([3], pp. 44)

With these simplification, the resulting trajectory affixed with the velocity at certain points are shown in fig. 7.18. We set up the coordinate system such that its origin coincides with the initial point. The axes directions are chosen to point in the horizontal and vertical directions, lining up the y-direction opposite to the gravitational acceleration direction. Therefore, we may write

$$a_x = 0, \quad a_y = -g$$

showing the independency of the x- and y-motions. Consequently, we may integrate the acceleration to obtain the corresponding velocity and positional components as

$$a_x = 0, \quad v_x = v_{xo}, \quad x = x_o + v_{xo}t$$
$$a_y = -g, \quad v_y = v_{y_o} - gt, \quad y = y_o + v_{y_o}t - \frac{1}{2}gt^2$$

for the given initial position and velocity components of x_o, y_o, v_{xo}, and v_{y_o}. Further, we may apply the acceleration-displacement relationship in the y-direction to derive for the following:

$$v_y^2 = {v_{y_o}}^2 - 2g\left(y - y_o\right)$$

To gain further understanding in velocity variation, we evaluate and draw the associated velocity and its components at certain points. It is easily observed that the velocity component in x-direction remains unchanged and equal to the initial value due to zero acceleration component in this direction. However, the magnitude of the velocity component in y-direction will gradually decrease until it becomes zero at the apex. After that, it will gradually increase in the opposite direction. This causes the path to turn back toward the ground. One might try to eliminate the intermediate time variable t in the parametric equations of $x(t)$ and $y(t)$ to verify that the trajectory is indeed described by the parabolic equation.

Example 7.8 ([4], Prob. 2/81) A particle is ejected from the tube at A with a velocity v at an angle θ with the vertical y-axis. A strong horizontal wind gives the particle a constant horizontal acceleration a in the x-direction. If the particle strikes the ground at a point directly under its released position, determine the height h of point A. The downward y-acceleration may be taken as the constant g.

Solution: The particle is subject to independent mutually perpendicular acceleration field $\overline{a}$ and $\overline{g}$. Hence we choose to set up the rectangular coordinate frame that has their axes aligned with the positive acceleration directions. The acceleration acting on the particle is shown in fig. 7.20.

With the chosen coordinate frame, the initial position and velocity are

$$v_{xo} = -v\sin\theta, \quad x_o = 0$$

$$v_{y_o} = v\cos\theta, \quad y_o = 0$$

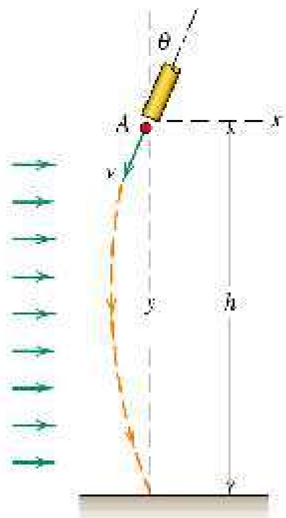


Figure 7.19: Example 7.8 ([4], pp. 50)

We then integrate the acceleration relationship to obtain the velocity and position as a function of time:

$$a_x = a, \quad v_x = -v\sin\theta + at, \quad x = -vt\sin\theta + \frac{1}{2}at^2$$

$$a_y = g, \quad v_y = v\cos\theta + gt, \quad y = vt\cos\theta + \frac{1}{2}gt^2$$

At the point where the particle hits the ground, $x = 0$ and $y = h$. Using the $x(t)$ relationship, we may solve for the hitting time

$$0 = t\left(\frac{at}{2} - v\sin\theta\right), \quad t = \frac{2v\sin\theta}{a}$$

Substituting the time t into $y(t)$ relationship, the height h of point A may then be determined:

$$h = \frac{2v^2}{a}\sin\theta\left(\cos\theta + \frac{g}{a}\sin\theta\right)$$

Example 7.9 ([3], Prob. 2/75) Electrons are emitted at A with a velocity u at the angle θ into the space between two charged plates. The electric field between the plates is in the direction E and repels the electrons approaching the upper plate. The field produces an acceleration of the electrons in the E-direction of eE/m, where e is the electron charge and m is its mass. Determine the field strength E that will permit the electrons to cross one-half of the gap between the plates. Also find the distance s.

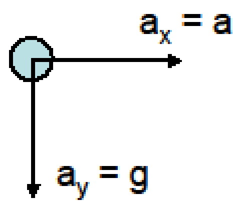


Figure 7.20: Solution to example 7.8

Solution: The electron traveling between two plates is subject to the electric field force of which its acceleration along the horizontal and vertical components may be written as

$$a_x = 0, \quad a_y = -\frac{eE}{m}$$

See fig. 7.22. The discharged electron will have the zero velocity component in y-direction when it reaches one-half of the gap. Therefore,

$[v_y dv_y = a_y dy]$

$$v_y^2 = {v_y}_o^{\ 2} + 2a_y (y - y_o)$$

$$0 = (u \sin\theta)^2 - 2\tfrac{eE}{m} \times \tfrac{b}{2}$$

The required field strength that permits the electron to cross one-half of the gap would follow the relationship that

$$E = \tfrac{mu^2}{eb} \sin^2\theta$$

When the electron falls back to the emitting plate, its coordinate values would be $(s, 0)$. Substitute these values into the x-coordinate relationship, we may solve for the time

$$x = s = ut\cos\theta, \quad t = \frac{s}{u\cos\theta}$$

Use this expression to eliminate the time in the y-coordinate relationship, the horizontal distance s may be determined.

$$0 = s\tan\theta\left(1 - \frac{s}{2b}\tan\theta\right), \quad s = 0 \ \text{ or } \ \frac{2b}{\tan\theta}$$

Example 7.10 ([4], Prob. 2/87) Water is ejected from the nozzle with a speed $v_o = 14$ m/s. For what value of the angle θ will the water land closest to the wall after clearing the top? Neglect the effects of wall thickness and air resistance. Where does the water land?

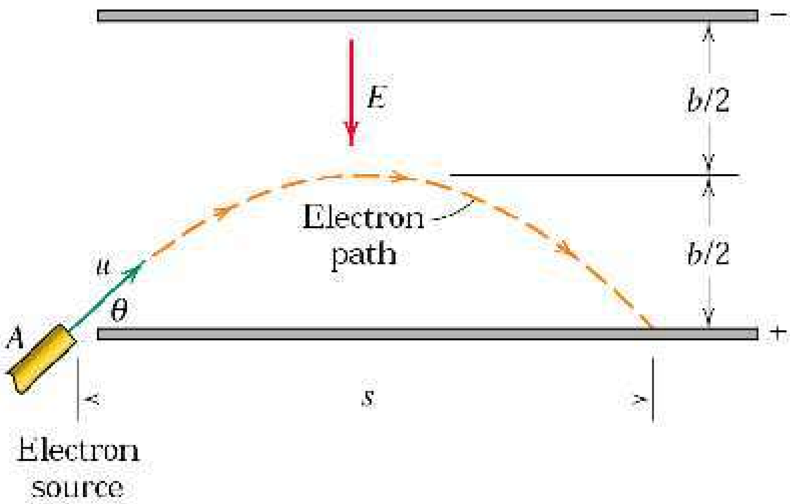


Figure 7.21: Example 7.9 ([3], pp. 50)

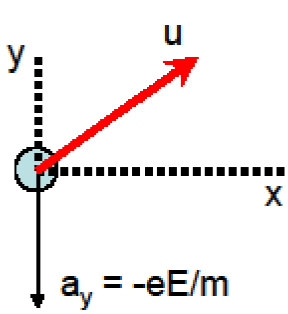


Figure 7.22: Solution to example 7.9

Solution: For this problem, we may treat a stream of water as a stream of particles. Set up the rectangular coordinate system having the origin at the point on the ground down below the nozzle A. At one particular particle just above the zero-thickness wall, the corresponding coordinate values are $(19, 1)$. Applying the x- and y-positional relationship with $a_x = 0$ and $a_y = -g$, the following equations hold:

$[x = v_{xo}t]$ $\qquad 19 = 14t\cos\theta$

$\left[y = y_o + v_{y_o}t + \frac{1}{2}a_y t^2\right]$ $\qquad 0.7 + \frac{1}{2}gt^2 = 14t\sin\theta$

By recognizing the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, square the above equations and sum them together, we can eliminate the unknown angle θ and solve the polynomial equation for the time t.

$$14^2t^2 = 19^2 + \left(0.7 + \frac{1}{2}gt^2\right)^2, \quad t = 2.14,\ 1.81 \quad \text{sec}$$

The corresponding nozzle angle may be solved for after substituting the time back into either one of the positional relationship.

$$\theta = 50.64^\circ,\ 41.43^\circ$$

Trajectory sketch of these two solutions is drawn in fig. 7.23. It is seen that path (1) with the nozzle angle of 50.64° will make the water jet land closest to the wall. For the coordinate frame employed, coordinate values of the place where the water lands are $(x, 0)$ where x is the horizontal distance measured from the nozzle A. The y-coordinate relation will allow us to solve for the spent time

$$0 = 0.3 + 14t\sin 50.64 - \frac{1}{2}gt^2, \quad t = 2.234 \quad \text{sec}$$

Then, the associated x-coordinate value would be

$$x = 14t\cos 50.64 = 19.835 \quad \text{m}$$

Hence the water lands at $19.835 - 19 = 0.835$ m to the right of the wall.

Example 7.11 ([3], Prob. 2/97) A projectile is ejected into an experimental fluid at time $t = 0$. The initial speed is v_o and the angle to the horizontal is θ. The drag on the projectile results in an acceleration term $\overline{a}_D = -k\overline{v}$, where k is a constant and $\overline{v}$ is the velocity of the projectile. Determine the x- and y-components of both the velocity and displacement as functions of time. What is the terminal velocity? Include the effects of gravitational acceleration.

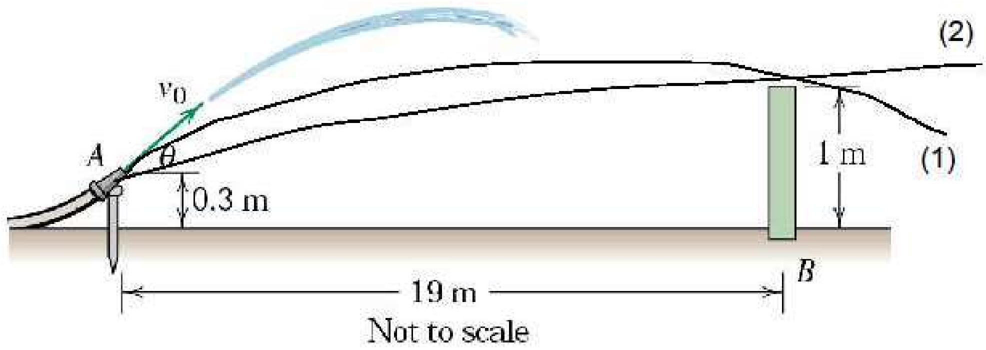


Figure 7.23: Example 7.10 ([4], pp. 51)

Solution: Two forces act on the bullet, namely the fluid drag force and the gravitational force. The drag force produces the deceleration that is proportional to the velocity which, for the given coordinate frame, we may write

$$\overline{a} = -k\overline{v} - g\mathbf{j} = (-kv_x)\,\mathbf{i} + (-kv_y - g)\,\mathbf{j}$$

Determination of the velocity and displacement is quite straightforward as before. The initial speed v_o and its direction θ measured against the x-axis are given. The bullet starts at the entrance port, for which we have set up the coordinate values as $(0, 0)$. The only difficulty is the acceleration is not constant, and hence we need to use some integral formula. For the x-component,

$\left[a_x = \frac{dv_x}{dt}\right]$
$$\int_{v_o \cos\theta}^{v_x} \frac{1}{-kv_x} dv_x = \int_0^t dt$$

$$v_x = (v_o \cos\theta)\, e^{-kt}$$

$\left[v_x = \frac{dx}{dt}\right]$
$$\int_0^t (v_o \cos\theta)\, e^{-kt} dt = \int_0^x dx$$

$$x = \frac{v_o \cos\theta}{k}\left(1 - e^{-kt}\right)$$

For the y-component,

$\left[a_y = \frac{dv_y}{dt}\right]$
$$\int_{v_o \sin\theta}^{v_y} \frac{1}{-kv_y - g} dv_y = \int_0^t dt$$

$$v_y = \left(v_o \sin\theta + \frac{g}{k}\right) e^{-kt} - \frac{g}{k}$$

$\left[v_y = \frac{dy}{dt}\right]$
$$\int_0^t \left\{\left(v_o \sin\theta + \frac{g}{k}\right) e^{-kt} - \frac{g}{k}\right\} dt = \int_0^y dy$$

$$y = \frac{1}{k}\left(v_o \sin\theta + \frac{g}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t$$

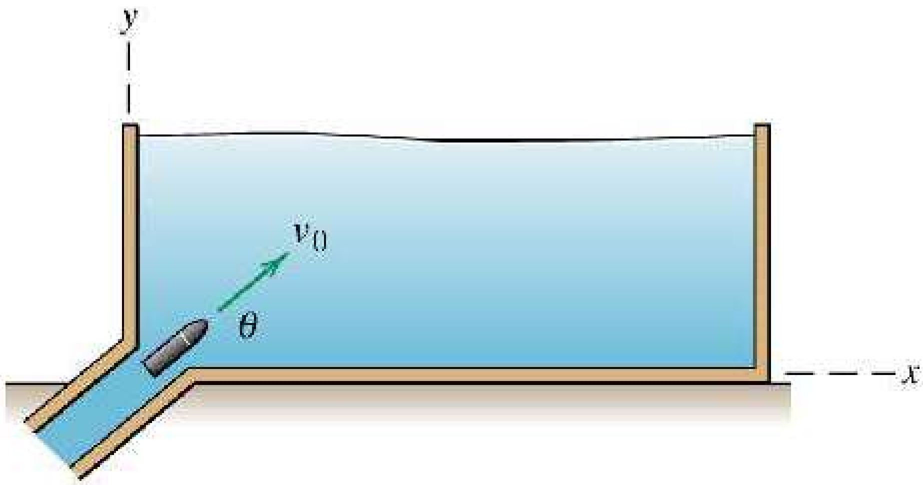


Figure 7.24: Example 7.11 ([3], pp. 54)

Terminal velocity is the velocity as $t \to \infty$. Take the limit as $t \to \infty$ in the expression for $v_x(t)$ and $v_y(t)$, we have

$$(v_x)_{t\infty} = 0, \quad (v_y)_{t\infty} = -\frac{g}{k}$$

Example 7.12 ([3], Prob. 2/100) A projectile is launched with speed v_o from point A. Determine the launch angle θ that results in the maximum range R up the incline of angle α (where $0 \le \alpha \le 90°$). Evaluate your results for $\alpha = 0, 30$, and 45°.

Solution: For the projectile motion under the gravity field, $a_x = 0$ and $a_y = -g$. Integrating them with respect to time and applying the initial velocity conditions, the velocity components would be

$$v_x = v_o \cos\theta, \quad v_y = v_o \sin\theta - gt$$

Then, we integrate the velocity expressions to determine the displacement along x- and y-directions as

$$x = v_o t \cos\theta, \quad y = v_o t \sin\theta - \frac{1}{2} g t^2$$

Consider the particle meeting the incline at B, we may use the x-coordinate expression to solve for the corresponding time as

$$R \cos\alpha = v_o t \cos\theta$$

$$t = \frac{R\cos\alpha}{v_o\cos\theta}$$

Substitute the time into the y-coordinate expression to construct the relation between R and θ;

$$R\sin\alpha = v_o\left(\frac{R\cos\alpha}{v_o\cos\theta}\right)\sin\theta - \frac{g}{2}\left(\frac{R\cos\alpha}{v_o\cos\theta}\right)^2$$

which, after some manipulation, may be reduced to

$$2v_o^2\cos^2\theta\tan\alpha = 2v_o^2\sin\theta\cos\theta - gR\cos\alpha$$

Since the change in the projectile launch angle θ causes the change in the inclined distance R, the maximum distance happens when $\frac{dR}{d\theta} = 0$. Therefore we may take the total differentiation on the previous relation,

$$-2v_o^2\tan\alpha\left(2\cos\theta\sin\theta d\theta\right) = v_o^2\left(2\cos 2\theta d\theta\right) - g\cos\alpha dR$$

and rearrange the expression

$$\frac{dR}{d\theta} = \frac{2v_o^2}{g\cos\alpha}\left(\cos 2\theta + \sin 2\theta\tan alpha\right)$$

which will be equated to zero, to solve for the optimal angle θ^*. Consequently,

$$\tan 2\theta^* = -\frac{1}{\tan\alpha}$$

$$2\theta^* = \tan^{-1}\left(-\frac{1}{\tan\alpha}\right) = 180^\circ - \tan^{-1}\left(\frac{1}{\tan\alpha}\right) = 180^\circ - (90^\circ - \alpha) = 90^\circ + \alpha$$

or

$$\theta^* = \frac{90^\circ + \alpha}{2}$$

Example 7.13 ([4], Prob. 2/95) Determine the equation for the envelope a of the parabolic trajectories of a projectile fired at any angle but with a fixed muzzle velocity u. (*Hint*: Substitute $m = \tan\theta$, where θ is the firing angle with the horizontal, into the equation of the trajectory. The two roots m_1 and m_2 of the equation written as a quadratic in m give the two firing angles for the two trajectories shown such that the shells pass through the same point A. Point A will approach the envelope a as the two roots approach equality.) Neglect air resistance and assuming g is constant.

Solution: We shall start from the acceleration of x- and y-components to determine their corresponding velocity and displacments.

$$a_x = 0, \quad v_x = u\cos\theta, \quad x = ut\cos\theta$$

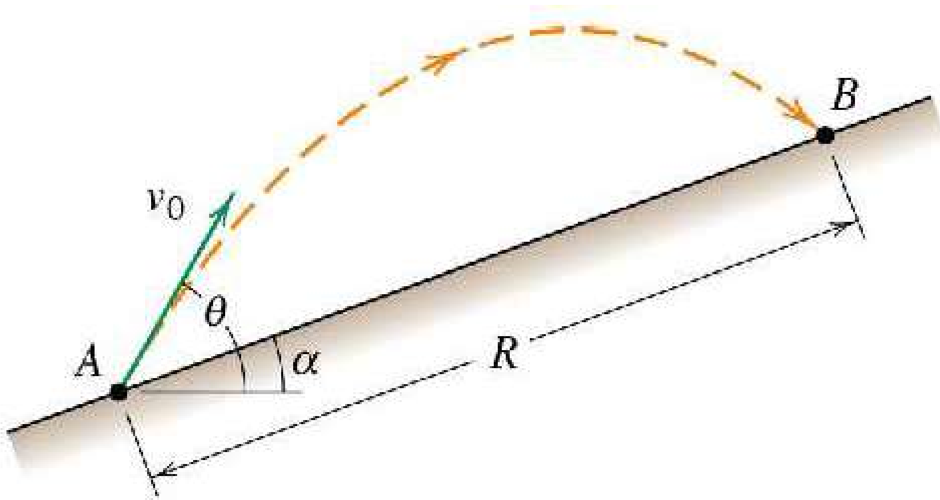


Figure 7.25: Example 7.12 ([3], pp. 54)

$$a_y = -g, \quad v_y = u \sin\theta - gt, \quad y = ut \sin\theta - \frac{1}{2} g t^2$$

for a particular firing angle θ and velocity u. Next, we should determine the projectile trajectory by eliminating the common time parameter t. The trajectory equation is then

$$y = x \tan\theta - \frac{g}{2} \left(\frac{x^2}{u^2 \cos^2\theta} \right)$$

which confirm the parabolic shape of the path.

Suppose we would like the trajectory to pass through a point A that has the coordinate (x, y) by varying the firing angle solely. The required angle may be determined from the above trajectory equation. The angle θ appears as the variable of two transcendental functions, namely $\tan\theta$ and $\cos\theta$. Both are related by the following trigonometric identity:

$$1 + \tan^2\theta = \sec^2\theta = \frac{1}{\cos^2\theta}$$

Let $m = \tan\theta$. Hence $\frac{1}{\cos^2\theta} = 1 + m^2$. Make use of these new expression with the trajectory equation, we may rearrange and rewrite it as

$$g x^2 m^2 - 2 x u^2 m + \left(2 y u^2 + g x^2 \right) = 0$$

a quadratic function of the variable m. Hence there are two roots according to the fact that point A may be reached from two distinct paths as seen in fig. 7.26. Interestingly, as those two paths approach each other, the point (x, y) will become a point on the envelope of the family of trajectories. Consequently,

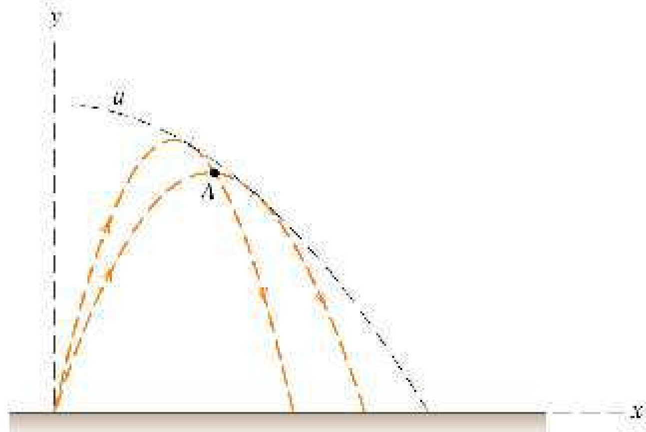


Figure 7.26: Example 7.13 ([4], pp. 53)

the two distinct roots must become two *repeated roots*. This calls for the zero discriminant. Specifically,

$$\left(2xu^2\right)^2 - 4gx^2\left(2yu^2 + gx^2\right) = 0$$

In other words, the following relation must be hold for any point (x, y) *on the envelope*

$$y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$$

That is, it is the envelope equation.

7.5 Normal and Tangential Coordinates $(n-t)$

The normal and tangential coordinate system is quite different from the rectangular one we have studied so far in a way that it is intrinsically a *moving coordinate system* (any coordinate system may not be fixed, though). The coordinate frame is attached with the particle and hence *moves and rotates* with it along the path. See fig. 7.27.

To set up the n-t coordinate frame, we place the frame origin right at the particle. The positive normal, denoted as n, direction will point toward the (local) center of curvature. In turn, the tangential, denoted as t, axis is perpendicular to the n-axis. Its positive direction may be selected arbitrarily, although commonly it points in the same way as the particle moves along the path. The unit vectors associated with the n and t axes are $\mathbf{e}_n$ and $\mathbf{e}_t$ respectively.

We are now ready to analyze fundamental kinematic parameters represented in the n-t coordinate frame. Position of the particle at any time is obviously

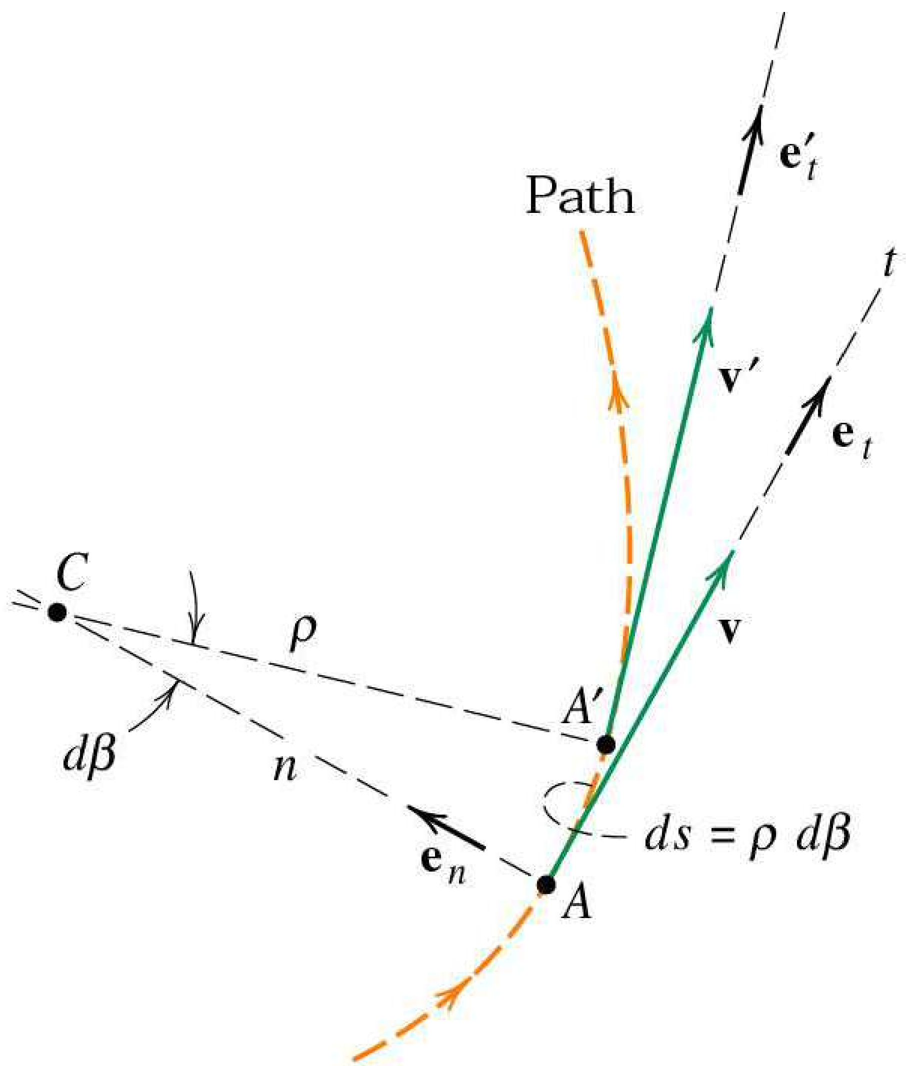


Figure 7.27: Successive n-t coordinate frames showing the instantaneous center of curvature ([3], pp. 55)

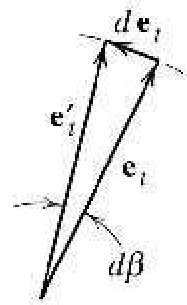


Figure 7.28: Change in $\mathbf{e}_t$ by the infinitesimal rotation ([3], pp. 55)

$0\mathbf{e}_n + 0\mathbf{e}_t$. For the velocity, one may be tempted to take the derivative of the position vector with respect to time. This would give us the null velocity, which is quite surprising since no matter how the particle actually moves, the obtained velocity would always be zero. What went wrong is that the position is the *relative position*. In other words, we measure the position of the particle with respect to the moving frame. The resulting velocity would then be the relative velocity. An observer moving with the n-t frame would see the particle not moving.

Nevertheless, we will resort to considering fig. 7.27 to determine the absolute velocity. At a moment, the particle is at A. After time elapsed, it moves to A' along the curved path. As the elapsed time converges to zero, point A and A' becomes coincident. The displacement becomes smaller and approach $ds = \rho d\beta$, where ρ is the radius of curvature of the path and $d\beta$ is the sweeping angle the radius of curvature traveled during that small interval. Taking the limit of the displacement as time interval converges to zero is equivalent to differentiating the displacement with respect to time, which is, according to the definition, the speed. Therefore,

$$v = \frac{ds}{dt} = \frac{\rho d\beta}{dt} = \rho\dot{\beta}$$

Since the n-t coordinate frame has the n-axis pointing toward the center of curvature and perpendicular to the trajectory, it is implied that the t-axis, which is tangent to the path, will be in the direction of the velocity vector. Therefore, the velocity vector may be expressed as

$$\overline{v} = v\mathbf{e}_t = \rho\dot{\beta}\mathbf{e}_t \tag{7.19}$$

That is, the velocity points along the t-axis. As the particle travels along $+t$-direction, $\dot{\beta} > 0$.

Differentiating the (absolute) velocity to determine the acceleration, we have

$$\overline{a} = \frac{d\overline{v}}{dt} = v\dot{\mathbf{e}}_t + \dot{v}\mathbf{e}_t$$

which can be further simplified into fundamental terms. Consider the change in $\mathbf{e}_t$, which is a free unit vector. Therefore, the change will get involved with the change in its direction only. Figure 7.28 illustrates $\mathbf{e}_t$ being rotated by the small angle $d\beta$ to a new vector $\mathbf{e}_t^{'}$. Difference between these two vectors is the vector $d\mathbf{e}_t$:

$$d\mathbf{e}_t = |\mathbf{e}_t|\, d\beta\mathbf{e}_n = d\beta\mathbf{e}_n$$

Taking the limit as time duration approaches zero, we have

$$\dot{\mathbf{e}}_t = \dot{\beta}\mathbf{e}_n \tag{7.20}$$

As a corollary, we also have

$$\dot{\mathbf{e}}_n = -\dot{\beta}\mathbf{e}_t \tag{7.21}$$

With these relationships, we may decompose the acceleration into n- and t-components as

$$\overline{a} = v\dot{\beta}\mathbf{e}_n + \dot{v}\mathbf{e}_t \tag{7.22}$$

Further elaboration might be handy:

$$a_n = v\dot{\beta} = \rho\dot{\beta}^2 = \frac{v^2}{\rho}$$

$$a_t = \dot{v} = \ddot{s} = \rho\ddot{\beta} + \dot{\rho}\dot{\beta}$$

From the last relationship, one may observe that the tangential acceleration behaves as if it is the rectilinear acceleration. That is it is the time derivative of the velocity, or the second time derivative of the displacement. We can extend our perspective on rectilinear motion to the curvilinear motion as long as the related components are all in tangential direction. The normal direction quantities are there to make the path deviate from the straight line path. Therefore the following relation holds as well:

$$vdv = a_t ds$$

Consider fig. 7.29 which shows the change in velocity and its components. As time duration approaches zero, $\overline{a}_n$ indicates the change in the direction of $\overline{v}$ while $\overline{a}_t$ indicates the change in the magnitude. Note that the component a_n will always direct toward the center of curvature. The component a_t, however, will be in $+t$-direction if the speed v is increasing. It may be in $-t$-direction had the speed is decreasing. Consequently though the acceleration direction does not lie in either one pricipal direction solely as for the velocity case (in the tangential direction), it will tend to the side (divided by the curvature tangent) where the center of curvature lives.

A final comment is that the kinematics quantities are intrinsic. Even they may be represented in some specific coordinate frame, the inherent property such

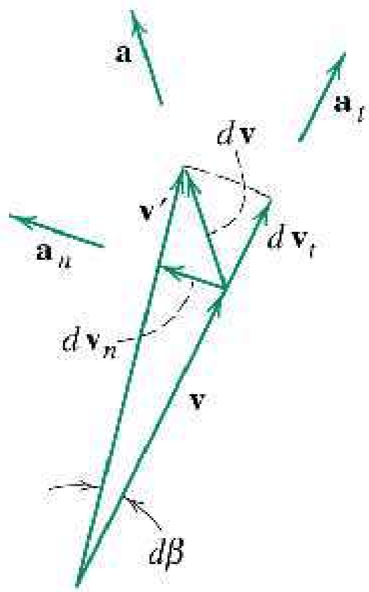


Figure 7.29: Infinitesimal change in the velocity and the relationship with the acceleration represented in n-t coordinate system ([3], pp. 55)

as the magnitude and the direction must not be changed. For the problem using the x-y and the n-t coordinate frames, the following may be stated:

$$\overline{v} = v_x\mathbf{i} + v_y\mathbf{j} = v\mathbf{e}_t$$

$$\overline{a} = a_x\mathbf{i} + a_y\mathbf{j} = a_n\mathbf{e}_n + a_t\mathbf{e}_t$$

Example 7.14 ([3], Prob. 2/122) The camshaft drive system of a four-cylinder automobile engine is shown. As the engine is revved up, the belt speed v changes uniformly from 3 m/s to 6 m/s over a 2 second interval. Calculate the magnitudes of the accelerations of point P_1 and P_2 half way through this time interval.

Solution: Since the timing belt velocity is increased uniformly, the (tangential) acceleration is constant. Therefore

$$a_t = \frac{dv}{dt} = \frac{\Delta v}{\Delta t} = \frac{6-3}{2} = 1.5 \quad \text{m/s}^2$$

This constant acceleration also implies that, at the time half of the interval, the belt speed is

$$v = \frac{3+6}{2} = 4.5 \quad \text{m/s}$$

Point $P1$ is moving in circular path around the sprocket. Therefore it will have both the tangential and normal components of the acceleration. The

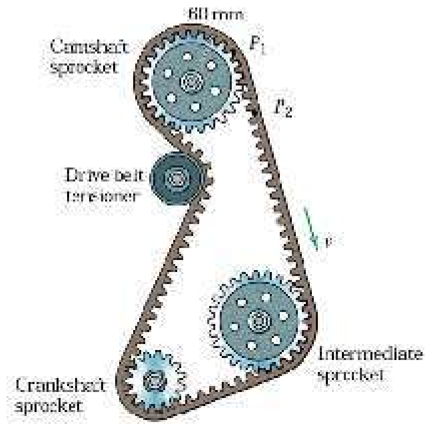


Figure 7.30: Example 7.14 ([3], pp. 64)

normal acceleration may be calculated as

$[a_n = v^2/\rho]$ $$a_n = \frac{4.5^2}{0.06} = 337.5 \ \ \text{m/s}^2$$

Therefore the acceleration of $P1$ becomes

$$a_{P1} = \sqrt{a_n^2 + a_t^2} = \sqrt{337.5^2 + 1.5^2} = 337.5 \ \ \text{m/s}^2$$

For point $P2$, it is moving along the tangent between the camshaft and the intermediate sprocket. Hence the path is straight line and so the acceleration is just the tangential acceleration of the belt.

$$a_{P2} = a_t = 1.5 \ \ \text{m/s}^2$$

Example 7.15 ([3], Prob. 2/126) A baseball player releases a ball with the initial conditions shown in the figure. Determine the radius of curvature of the trajectory a) just after release and b) at the apex. For each case, compute the time rate of change of the speed.

Solution: We must use the *n*-*t* coordinate system since the problem asks for the radius of curvature ρ. Refer to the velocity and acceleration sketches shown in fig. 7.32. At just after release, the normal acceleration component is

$$a_n = g \cos 30$$

because the throwing angle is 30° up with respect to the horizontal line. From this, we may determine the radius of curvature

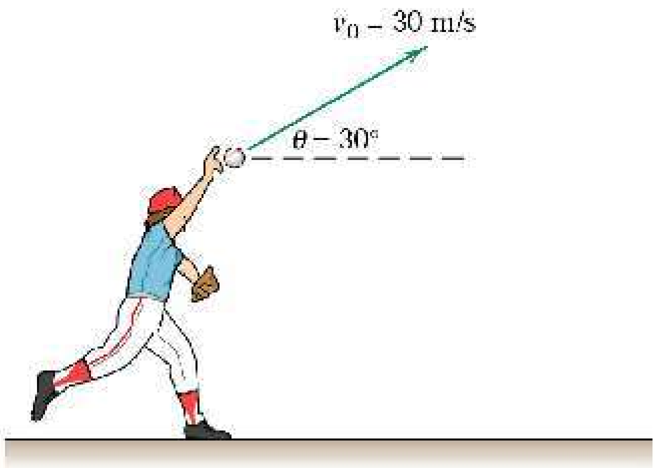


Figure 7.31: Example 7.15 ([3], pp. 65)

$[a_n = v^2/\rho]$ $\qquad g\cos 30 = \frac{30^2}{\rho}, \quad \rho = 105.9 \text{ m}$

The time rate of change of the speed is the tangential acceleration, which is obvious from the sketch as

$$\dot{v} = a_t = -g\sin 30 = -4.905 \ \ \text{m/s}^2$$

Same quantities for the baseball at the apex may be determined in the same manner. At the apex,

$$a_n = g = \frac{v^2}{\rho} = \frac{(30\cos 30)^2}{\rho}, \quad \rho = 68.8 \text{ m}$$

where the speed has been reduced to $30\cos 30$ m for the vertical component that now becomes null.

$$\dot{v} = a_t = 0 \ \ \text{m/s}^2$$

Example 7.16 ([4], Prob. 2/112) Pin P in the crank PO engages the horizontal slot in the guide C and controls its motion on the fixed vertical rod. Determine the velocity $\dot{y}$ and the acceleration $\ddot{y}$ of guide C for a given value of the angle θ if a) $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$ and b) if $\dot{\theta} = 0$ and $\ddot{\theta} = \alpha$.

Solution: Guide C will move up or down according to the motion of the rotating crank PO. By the mechanical constraint that the pin P must always be in the horizontal slot, their vertical motion must be identical. From the given rotation of the crank, we may deduce the velocity and the acceleration at P as

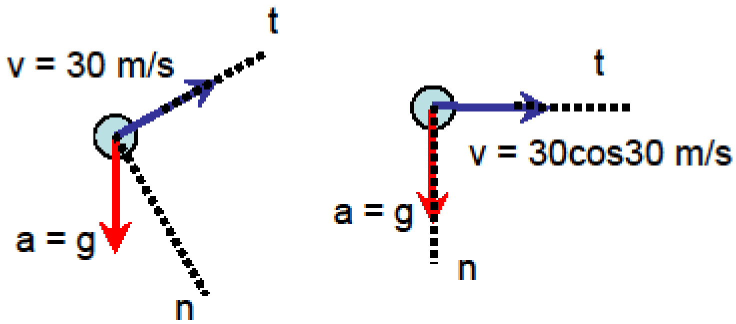


Figure 7.32: Solution to example 7.15

$[v = r\omega]$ $$v = r\omega$$

$[a_n = v^2/r]$ $$a_n = r\omega^2$$

$[a_t = r\alpha]$ $$a_t = 0$$

for case a) where $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$. The motion of P for case b) where $\dot{\theta} = 0$ and $\ddot{\theta} = \alpha$ may be computed similarly as

$$v = 0$$

$$a_n = 0$$

$$a_t = r\alpha$$

As explained already, the projection of the velocity and the acceleration in the vertical direction yield their counterparts of the guide. With the kinematic vectors drawn in fig. 7.34, we would have

$$\dot{y} = r\omega\sin\theta, \quad \ddot{y} = r\omega^2\cos\theta$$

for case a), and

$$\dot{y} = 0, \quad \ddot{y} = r\alpha\sin\theta$$

for case b).

Example 7.17 ([4], Prob. 2/129) The pin P is constrained to move in the slotted guides that move at right angles to one another. At the instant represented, A has a velocity to the right of 0.2 m/s which is decreasing at the rate of 0.75 m/s each second. At the same time, B is moving down with a velocity of 0.15

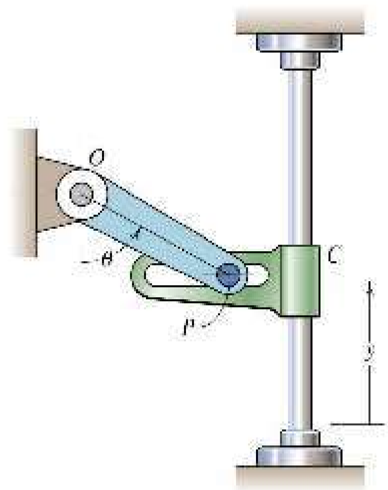


Figure 7.33: Example 7.16 ([4], pp. 62)

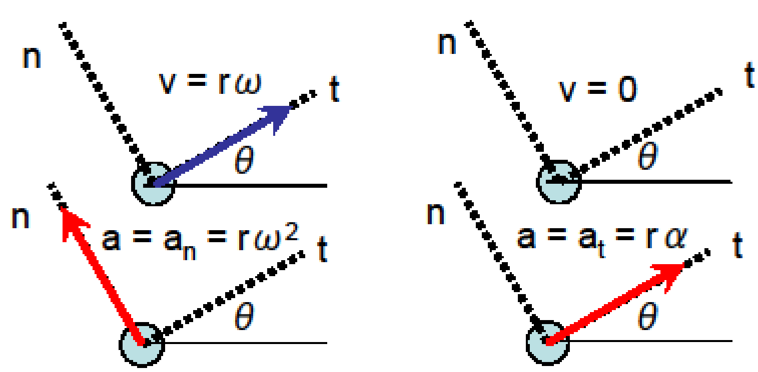


Figure 7.34: Solution to example 7.16

m/s which is decreasing at the rate of 0.5 m/s each second. For this instant, determine the radius of curvature ρ of the path followed by P. Is it possible to determine also the time rate of change of ρ.

Solution: Information about motion of pin P is described by the motion of the slotted guides moving perpendicular to each other. Therefore the motion provided is expressed implicitly in the x-y coordinate frame. We must transform this description into the n-t coordinate frame so the radius of curvature may be determined. Since guides A and B are moving relatively at the right angle, their motion are independent. Additionally, their motions are imparted to pin P. Therefore its velocity and acceleration, representing in the x-y coordinate frame depicted in fig. 7.36, are

$$\overline{v} = 0.2\mathbf{i} - 0.15\mathbf{j}$$

$$\overline{a} = -0.75\mathbf{i} + 0.5\mathbf{j}$$

Recalling the basic fact that velocity vector lies along the t-axis, we are able to determine its direction as shown in fig. 7.36. Then, we will be able to determine the direction of the n-axis from the direction of the t-axis and the acceleration. That is, it will be making right angle to the t-axis, where the positive direction will be pointing to the side where the direction of the acceleration resides. As a result, we will be able to completely indicate the n-t coordinate frame. See fig. 7.36.

Knowing the orientation of the frame n-t with respect to frame x-y, the unit vector $\mathbf{e}_t$ may be expressed as

$$\mathbf{e}_t = 0.8\mathbf{i} - 0.6\mathbf{j}$$

Hence the acceleration components in the normal and tangential directions may be determined as follow:

$$\overline{a}_t = (\overline{a} \cdot \mathbf{e}_t)\,\mathbf{e}_t = -0.72\mathbf{i} + 0.54\mathbf{j}$$

$$\overline{a}_n = \overline{a} - \overline{a}_t = -0.03\mathbf{i} - 0.04\mathbf{j}$$

Now we may determine the radius of curvature by

$[a_n = v^2/\rho]$ $$\rho = \tfrac{0.25^2}{0.05} = 1.25 \text{ m}$$

For the question of determining $\dot{\rho}$, consider the following kinematic relation which contains this term;

$$a_t = \dot{v} = \rho\ddot{\beta} + \dot{\rho}\dot{\beta} = \rho\ddot{\beta} + \dot{\rho}\left(\frac{v}{\rho}\right)$$

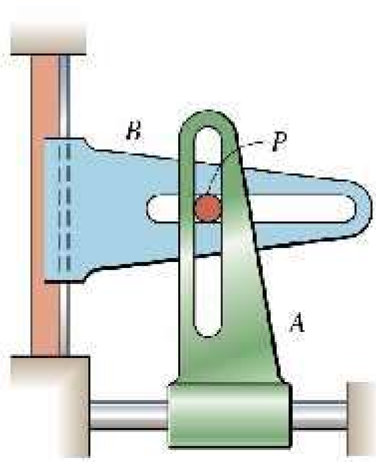


Figure 7.35: Example 7.17 ([4], pp. 66)

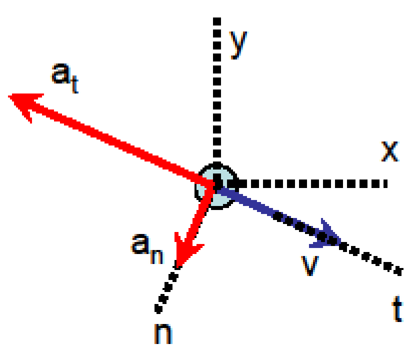


Figure 7.36: Solution to example 7.17

Currently, this equation has two unknowns, namely $\dot{\rho}$ (the one we need) and $\ddot{\beta}$. Therefore, $\dot{\rho}$ cannot be determined until $\ddot{\beta}$ is known.

Example 7.18 ([3], Prob. 2/134) As a handling test, a car is driven through the slalom course shown. It is assumed that the car path is sinusoidal and that the maximum lateral acceleration is $0.7g$. If the testers wish to design a slalom through which the maximum speed is 80 km/h, what cone spacing L should be used?

Solution: From the geometry of the path in fig. 7.37, we may write down its equation as

$$y = 3\sin\omega x = 3\sin\left(\frac{\pi}{L}x\right)$$

It is specified that the maximum lateral acceleration during the course is $a_n = 0.7g$. Since $a_n = v^2/\rho$, the value will be at peak if the velocity is maximum *and* the radius of curvature becomes minimum. From the sketched sinusiodal trajectory, the peak point of the waveform will have minimum radius of curvature. Associated with the maximum speed of 80 km/h, the minimum value would be

$$\rho_{\min} = \frac{v_{\max}^2}{(a_n)_{\max}} = \frac{(80\times 10/36)^2}{(0.7g)} = 71.9 \text{ m}$$

Now we need to think of some relations which will link the radius of curvature ρ to the unknown cone spacing L. Since L is contained in the trajectory equation, we may found the following relation from calculus to be helpful.

$$\rho = \frac{\left(1+y'^2\right)^{3/2}}{y''}$$

Differentiating the path equation above for y' and y'',

$$y' = 3\omega\cos\omega x, \quad y'' = -3\omega^2\sin\omega x$$

Then evaluate them at the peaks (minimum radius of curvature) where $\omega x = n\pi/2,\ n = \pm1, \pm3, ...$, we have

$$y' = 0, \quad y'' = \mp 3\omega^2$$

Substitute these derivatives into the x-y-frame radius of curvature formula, and solve for the desired L:

$$\mp 71.9 = \frac{1}{\mp 3\omega^2}$$

$$\omega = 0.0681 = \frac{\pi}{L} \quad \rightarrow \quad L = 46.14 \text{ m}$$

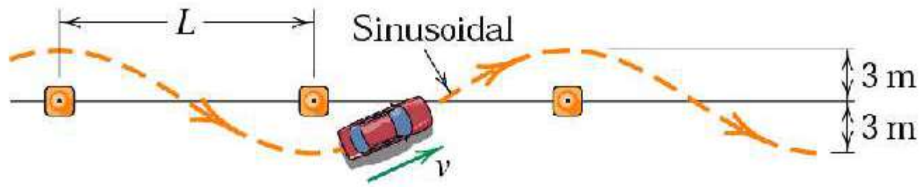


Figure 7.37: Example 7.18 ([3], pp. 67)

Example 7.19 A particle starts from rest at the origin and moves along the positive branch of the curve $y = 2x^{3/2}$ so that the distance s measured from the origin along the curve varies with the time t according to $s = 2t^3$, where x, y, and s are in millimeters and t is in seconds. Find the magnitude of the total acceleration of the particle when $t = 1$ s.

Solution: Recall that the curvilinear motion is the extension of the straight line motion with the normal acceleration added. Therefore along the tangential direction, we have

$$s = 2t^3, \quad v = \dot{s} = 6t^2, \quad a_t = \dot{v} = 12t$$

To determine the total acceleration, we need the normal acceleration which may be calculated by

$$a_n = \frac{v^2}{\rho} \quad \& \quad \rho = \frac{\left(1 + y'^2\right)^{3/2}}{y''}$$

For the specified path equation of

$$y = 2x^{3/2}$$

we have

$$y' = 3\sqrt{x} \quad \& \quad y'' = \frac{3}{2\sqrt{x}}$$

Therefore, the coordinate x, i.e. the location where we would like to evaluate the radius of curvature, must be known first.

However the elapsed time is given instead. Consequently, the relationship between the traveling distance, coordinates, and time must be established. Relation between distance traveled and time are specified to be $s = 2t^3$. When $t = 1$ second, the traveled distance is 2 mm.

The link between the distance traveled and its coordinates may be determined from the differential geometry as

$$\int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{1 + (y')^2} dx$$

Substitute $y'(x)$ for this particular path into the above relation and solve for x, we then have

$$2 = \int_0^x \sqrt{1+9x}dx$$

$$x = 0.913 \text{ mm}, \quad y = 1.746 \text{ mm}$$

Now we are ready to determine the total acceleration at $t = 1$ second. From the above equations, at that time, $v = 6$ mm/s and $a_t = 12$ mm/s^2. From the path integration, $x = 0.913$ mm, $y' = 2.8665$, $y'' = 1.57$, and so $\rho = 17.8$ mm. The normal acceleration now can be determined as

$$a_n = \frac{6^2}{17.8} = 2.02 \ \ \text{mm/s}^2$$

Consequently the total acceleration becomes

$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{2.02^2 + 12^2} = 12.17 \ \ \text{mm/s}^2$$

7.6 Polar Coordinates $(r-\theta)$

Another intrinsically moving coordinate system is the polar coordinate system. It is, in some respect, comparable to the previous normal and tangential coordinate system. Nevertheless, it has its own use especially in navigation system. In fact, the polar coordinate system is a degenerated version of the three dimensional spherical coordinate system.

Refer to fig. 7.38. To set up the polar, or r-θ, coordinate frame, it is a prerequisite to decide the reference origin and the reference direction. In the figure, we employ the x-y coordinate frame, where its origin and the x-axis act as the references. Suppose the particle is currently at point A. The origin of the r-θ frame is located right at the particle. The positive radial direction, denoted as r, lines up with the line connecting the reference origin and the r-θ frame origin. Then, the θ-axis will be perpendicular to the radial direction, of which its positive direction is selected to be in the same way as the particle moves along the path. The unit vectors associated with the r and θ axes are $\mathbf{e}_r$ and $\mathbf{e}_\theta$ respectively.

We are now ready to analyze fundamental kinematic parameters represented in the r-θ coordinate frame. Contrary to the n-t coordinate frame, position of the particle is not zero because we have set up additional references for the *absolute location* of the particle. The particle is located by the *radial distance* r from the origin and by the *angle* θ to the reference direction. As a result, the *absolute position* of the particle may be described by

$$\overline{r} = r\mathbf{e}_t \tag{7.23}$$

Note that the angle θ is implicitly used to fixate the direction of $\mathbf{e}_r$.

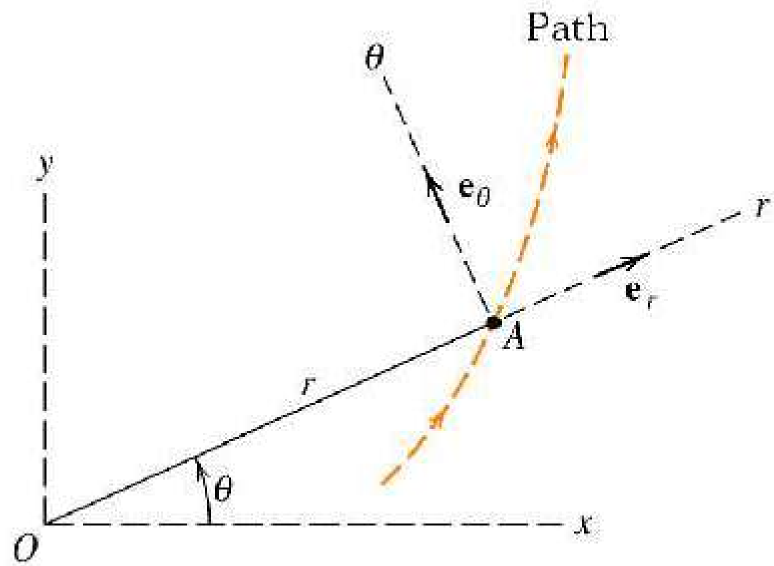


Figure 7.38: r-θ coordinate frame attached to the particle and the x-y reference coordinate frame ([3], pp. 68)

Since this is the absolute position, the velocity may be determined by differentiating the above expression to obtain

$$\overline{v} = \dot{\overline{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$$

which can be further resolved into fundamental terms. Consider the change in $\mathbf{e}_r$, which is a free unit vector. Therefore, the change of the vector will come from the change in its direction only. Figure 7.39 illustrates $\mathbf{e}_r$ (and $\mathbf{e}_\theta$) being rotated by the small angle $d\theta$ to a new vector $\mathbf{e}_r^{'}$. Difference between these two vectors is the vector $d\mathbf{e}_r$:

$$d\mathbf{e}_r = |\mathbf{e}_r|\, d\theta\mathbf{e}_\theta = d\theta\mathbf{e}_\theta$$

Taking the limit as time duration approaches zero, we have

$$\dot{\mathbf{e}}_r = \dot{\theta}\dot{\mathbf{e}}_\theta \tag{7.24}$$

As a corollary, we also have

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\dot{\mathbf{e}}_r \tag{7.25}$$

With these relationships, we may decompose the velocity into r- and θ-components as

$$\overline{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{7.26}$$

A glimpse on the equation may give us physical understanding of what's going on. The r-component of the velocity, $v_r = \dot{r}$, involves the change of the radial

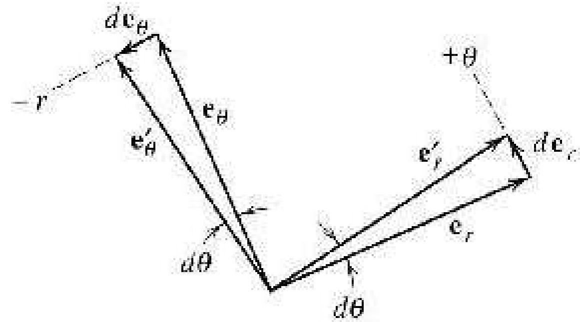


Figure 7.39: Change in $\mathbf{e}_r$ and $\mathbf{e}_\theta$ by the infinitesimal rotation ([3], pp. 68)

distance. Hence it shows how fast the position vector $\overline{r}$ contracts or stretches. About the θ-component, $v_\theta = r\dot{\theta}$ will be non-zero if the position vector is rotating. Therefore v_θ indicates the change of the position vector's direction. The magnitude of the velocity is readily calculated due to the orthogonality of the axes:

$$v = \sqrt{v_r^2 + v_\theta^2}$$

Differentiating the velocity to determine the acceleration and applying the expressions for the time rate of change of the unit vectors, we have

$$\overline{a} = \dot{\overline{v}} = (\ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r) + (\dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta)$$

$$\overline{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \tag{7.27}$$

One may reach the same expression on the acceleration by the graphical approach. Consider small changes in magnitude and direction of both velocity components depicted in fig. 7.40. Change in the magnitude of $\overline{v}_r$ is the vector $(d\dot{r})\mathbf{e}_r$. However change in the direction of $\overline{v}_r$ (caused by the small rotation $d\theta$) is the vector $(\dot{r}d\theta)\mathbf{e}_\theta$. In the same vein, change in the magnitude of $\overline{v}_\theta$ is the vector $\left(d(r\dot{\theta})\right)\mathbf{e}_\theta$. Lastly, change in the direction of $\overline{v}_\theta$ is the vector $(-r\dot{\theta}d\theta)\mathbf{e}_r$. As the time duration approaches zero, we may employ the definition of differentiation to conclude the same expression on the acceleration.

From this analysis, we will understand thoroughly the cause of each acceleration component. The r-component of the acceleration, $a_r = \ddot{r} - r\dot{\theta}^2$, indicates the change in magnitude of $\overline{v}_r$ and the change in direction of $\overline{v}_\theta$. About the θ-component, the change in direction of $\overline{v}_r$ or the change in magnitude of $\overline{v}_\theta$ will cause the non-zero accleration component $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d(r^2\dot{\theta})}{dt}$. Note the term $2\dot{r}\dot{\theta}$ in a_θ. It combines two effects of changing in both magnitude (of $\overline{v}_\theta$) and direction (of $\overline{v}_r$) of the velocity. The change in the radial distance and the angle

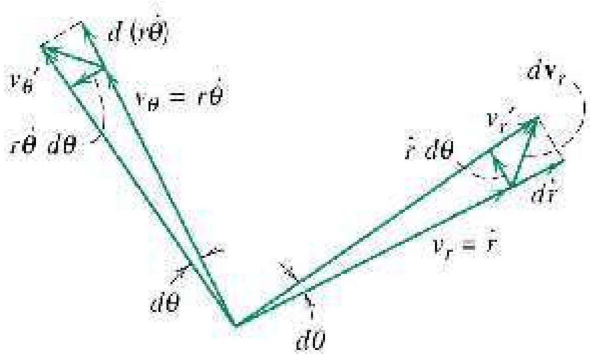


Figure 7.40: Infinitesimal change in the velocity and the relationship with the acceleration represented in r-θ coordinate system ([3], pp. 69)

must neither be null for this *Coriolis acceleration* to be non-zero. The magnitude of the acceleration is readily calculated due to the orthogonality of the axes:

$$a = \sqrt{a_r^2 + a_\theta^2}$$

One final note is that $a_r \neq \dot{v}_r$ and $a_\theta \neq \dot{v}_\theta$. This is because we must also account for change in the direction of both v_r and v_θ, which are $\dot{r}\dot{\theta}$ along $\mathbf{e}_\theta$ and $-r\dot{\theta}^2$ along $\mathbf{e}_r$, respectively.

Example 7.20 ([3], Prob. 2/147) The rocket is fired vertically and tracked by the radar shown. When θ reaches 60°, other corresponding measurements give the values $r = 9$ km, $\ddot{r} = 21$ m/s^2, and $\dot{\theta} = 0.02$ rad/s. Calculate the magnitudes of the velocity and acceleration of the rocket at this position.

Solution: At the position where $\theta = 60°$, the r-θ frame may be set up as shown in fig. 7.42. From the problem statement, the velocity and acceleration vectors point vertically upward. Therefore we may relate their geometric projections to the kinematic relationships as

$$v_\theta = v \sin 30 = r\dot{\theta} = 9000 \times 0.02 \quad \rightarrow \quad v = 360 \ \ \text{m/s}$$

$$a_r = a \cos 30 = \ddot{r} - r\dot{\theta}^2 = 21 - 9000 \times 0.02^2 \quad \rightarrow \quad a = 20.09 \ \ \text{m/s}^2$$

Example 7.21 ([3], Prob. 2/151) Link AB rotates through a limited range of the angle β, and its end A causes the slotted link AC to rotate also. For the instant represented where $\beta = 60°$ and $\dot{\beta} = 0.6$ rad/s constant, determine the corresponding values of $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$.

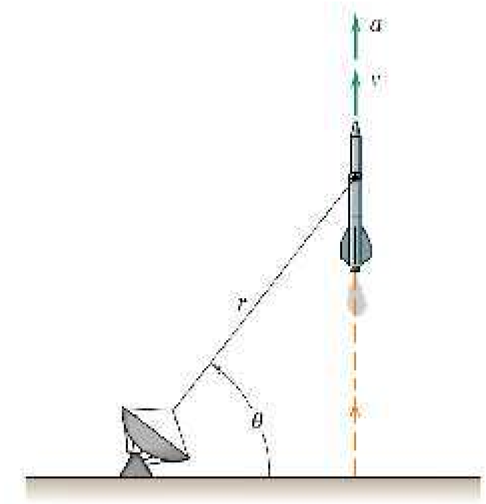


Figure 7.41: Example 7.20 ([3], pp. 75)

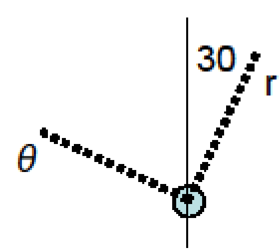


Figure 7.42: Solution to example 7.20

Solution: Motion of A is specified in terms of n-t coordinate system's parameters. The problem then asks for its description in r-θ coordinate system instead. Therefore, we first set up these two coordinate frames as depicted in fig. 7.44. Then we exploit the fundamental property of the kinematical parameters that are frame invariant. As a result, its velocity may be determined as

$$\left[v = \rho\dot{\beta}\right] \qquad v = 0.15 \times 0.6 = 0.09 \;\text{ m/s along } +t\text{-direction}$$

Since link AB rotates with constant angular velocity, $\ddot{\beta} = 0$ and hence $a_t = \rho\ddot{\beta} + \dot{\rho} + \dot{\beta} = 0$. Consequently, the acceleration of A comes from the normal acceleration solely.

$$\left[a_n = \rho\dot{\beta}^2\right] \qquad a = a_n = 0.15 \times 0.6^2 = 0.054 \;\text{ m/s}^2 \text{ along } +n\text{-direction}$$

These vectors will be equivalently represented in terms of r-θ coordinate system with its reference origin at C and horizontal reference line. At this instant, $r = 0.15$ m and $\theta = 60°$ by geometry calculation. With the aid of fig. 7.44, the velocity may be written in terms of its r- and θ-components as

$$\overline{v} = \overline{v}_r + \overline{v}_\theta = v\cos 30\mathbf{e}_r - v\sin 30\mathbf{e}_\theta = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

We may then solve for the time derivatives of the coordinates:

$$\dot{r} = 0.078 \;\text{ m/s}, \quad \dot{\theta} = -0.3 \;\text{ rad/s}$$

Similarly, we decompose the acceleration to the r- and θ-directions, and equate them to the expressions relating to r, θ, and their derivatives.

$$\overline{a} = \overline{a}_r + \overline{a}_\theta = -a\sin 30\mathbf{e}_r - a\cos 30\mathbf{e}_\theta = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

Hence,

$$\ddot{r} = -0.0135 \;\text{ m/s}^2, \quad \ddot{\theta} = 2.31 \times 10^{-4} \;\text{ rad/s}^2$$

Example 7.22 ([4], Prob. 2/150) The slotted arm OA forces the small pin to move in the fixed spiral guide defined by $r = K\theta$. Arm OA starts from rest at $\theta = \pi/4$ and has a constant counterclockwise angular acceleration $\ddot{\theta} = \alpha$. Determine the magnitude of the acceleration of the pin when $\theta = 3\pi/4$.

Solution: First of all, the pin P is constrained to move in the fixed spiral guide. This implies that the path of the pin described in r-θ coordinates shall be

$$r = K\theta$$

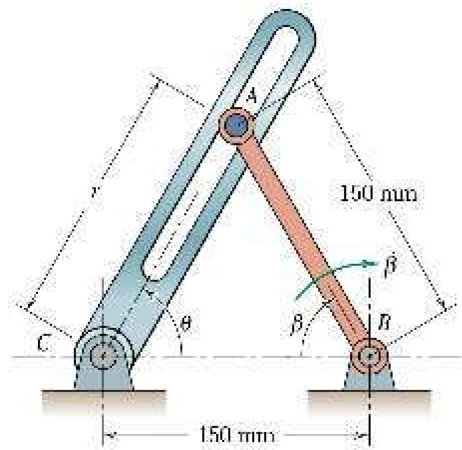


Figure 7.43: Example 7.21 ([3], pp. 76)

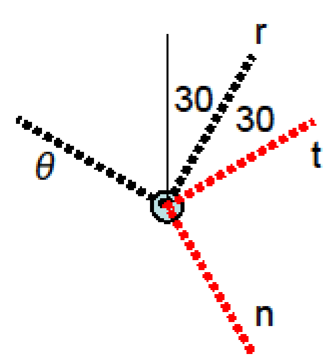


Figure 7.44: Solution to example 7.21

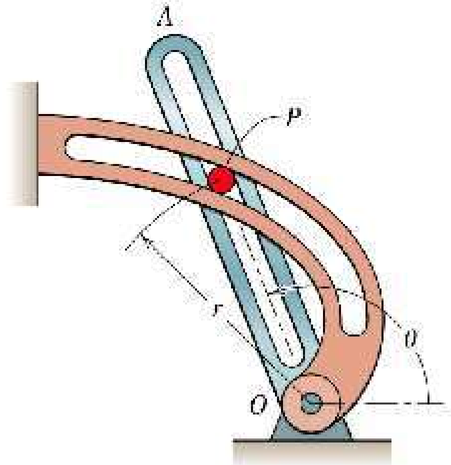


Figure 7.45: Example 7.22 ([4], pp. 76)

Because K is constant, the following velocity and acceleration equations result:

$$\dot{r} = K\dot{\theta}, \quad \ddot{r} = K\ddot{\theta}$$

Some information regarding to θ and its derivatives has been given. The motion lasts from $\theta = \pi/4$ to $\theta = 3\pi/4$, with the constant angular acceleration $\ddot{\theta} = \alpha$. With the motion start from rest, $\theta_o = 0$. Therefore we may be able to compute the ending angular velocity from

$$\left[\dot{\theta}^2 = \dot{\theta}_o^2 + 2\ddot{\theta}\left(\theta - \theta_o\right)\right] \qquad \dot{\theta}^2 = 2\alpha\left(\tfrac{3\pi}{4} - \tfrac{\pi}{4}\right) = \pi\alpha$$

The r-coordinate and its derivatives at the ending may be computed from the path equations shown above. The results are

$$r = 3K\pi/4, \quad \dot{r} = K\sqrt{\pi\alpha}, \quad \ddot{r} = K\alpha$$

All parameters related to calculating the acceleration are now determined. Therefore the acceleration of the pin when $\theta = 3\pi/4$ is

$$\overline{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

Substituting the parameter values and calculate for the magnitude, we have

$$a = 10.753K\alpha$$

Example 7.23 ([3], Prob. 2/160) The circular disc rotates about its center O with a constant angular velocity $\omega = \dot{\theta}$ and carries the two spring-loaded plungers shown. The distance b that each plunger protudes from the rim of the disc varies according to $b = b_o \sin 2\pi nt$, where b_o is the maximum protusion, n is the constant frequency of oscillation of the plungers in the radial slots, and t is the time. Determine the maximum magnitudes of the r- and θ-components of the acceleration of the ends A of the plungers during their motion.

Solution: We will apply the acceleration expression in determining the value at point A. To do so, we need to evaluate the r, θ parameters and their derivatives, for which we shall start from the position level. Position of the point A is conveniently described using (r, θ) coordinates. With the reference point O at the center point of the disc, the expression for the radial distance becomes

$$r = r_o + b = r_o + b_o \sin 2\pi nt$$

Differentiating it with respect to time, we have

$$\dot{r} = 2\pi n b_o \cos 2\pi nt, \quad \ddot{r} = -\left(2\pi n\right)^2 b_o \sin 2\pi nt$$

For θ-coordinate, the disc is rotating with constant angular velocity. Therefore,

$$\dot{\theta} = \omega, \quad \ddot{\theta} = 0$$

Substitute the above parameters into each acceleration component's expression:

$$a_r = \ddot{r} - r\dot{\theta}^2 = -r_o\omega^2 - \left(4\pi^2 n^2 + \omega^2\right) b_o \sin 2\pi nt$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 4\pi n\omega b_o \cos 2\pi nt$$

Their maximum magnitude may then be determined by recognizing the maximum value of sine or cosine function is 1. Hence

$$\left|a_r\right|_{\max} = r_o\omega^2 + \left(4\pi^2 n^2 + \omega^2\right) b_o$$

$$\left|a_\theta\right|_{\max} = 4\pi n\omega b_o$$

Example 7.24 ([3], Prob. 2/164) The small block P starts from rest at time $t = 0$ at point A and moves up the incline with constant acceleration a. Determine $\dot{r}$ and $\dot{\theta}$ as a function of time.

Solution: The block moves with constant acceleration allows us to apply the instant formula for the velocity and displacement. Because the block starts from rest, the displacement s measured from A follows

$$s = \frac{1}{2}at^2$$

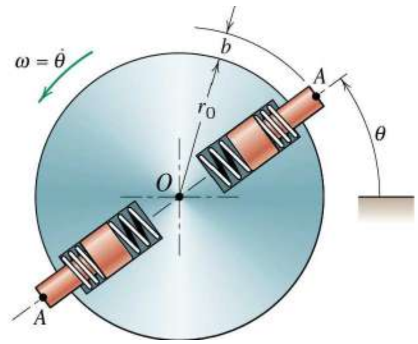


Figure 7.46: Example 7.23 ([3], pp. 78)

This motion described in (r, θ) coordinates may be achieved by considering the geometry of the relevant parameters in fig. 7.47. Accordingly, one may write

$$x = R + s\cos\alpha = R + \frac{at^2}{2}\cos\alpha, \quad \dot{x} = at\cos\alpha$$

$$y = s\sin\alpha = \frac{at^2}{2}\sin\alpha, \quad \dot{y} = at\sin\alpha$$

Therefore the radial distance indicating the position of block P relative to O, after simplification, is

$$r^2 = x^2 + y^2 = R^2 + Rat^2\cos\alpha + \frac{a^2t^4}{4}$$

Differentiating the relation with respect to time and solve for the required $\dot{r}$, we have

$$2r\dot{r} = 2Rat\cos\alpha + a^2t^3$$
$$\dot{r} = \frac{at\left(2R\cos\alpha + at^2\right)}{2\sqrt{R^2 + Rat^2\cos\alpha + \frac{a^2t^4}{4}}}$$

For determining $\dot{\theta}$, one may start from the following relation

$$\tan\theta = \frac{y}{x}$$

Differentiating the equation,

$$\dot{\theta}\sec^2\theta = \frac{x\dot{y} - y\dot{x}}{x^2}$$

Substituting the following relation

$$\sec^2\theta = \frac{1}{\cos^2\theta} = \frac{r^2}{x^2}$$

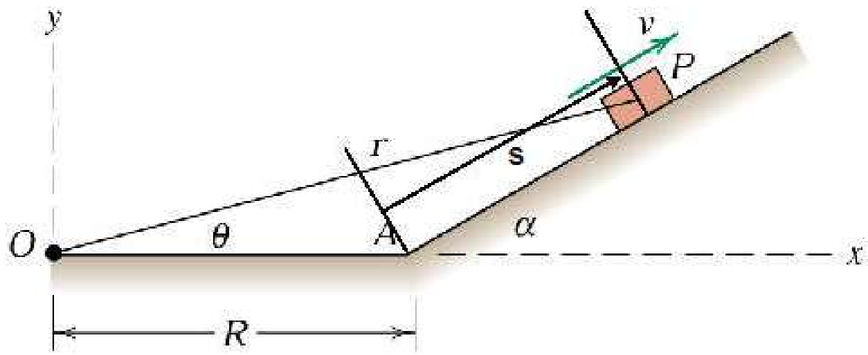


Figure 7.47: Example 7.24 ([3], pp. 79)

into the above equation, we finally have

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{Rat\sin\alpha}{R^2 + Rat^2\cos\alpha + \frac{a^2t^4}{4}}$$

Example 7.25 ([4], Prob. 2/159) The block P slides on the surface shown with constant speed $v = 0.6$ m/s and passes point O at time $t = 0$. If $R = 1.2$ m, determine the following quantities at time $t = 2\,(1 + \pi/3)$: r, θ, $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$.

Solution: The block slides with constant speed. Hence the distance traveled after $t = 2\,(1 + \pi/3)$ sec is

$$s = vt = 0.6 \times 2\,(1 + \pi/3) = 1.2 + 0.4\pi$$

Since the length of the horizontal floor is 1.2 m, the block moves up the quarter guide by 0.4π m, for which the corresponding position of the block may be located with the angle

$$\theta = \frac{0.4\pi}{1.2} = \frac{\pi}{3} = 60^\circ$$

measured from the vertical downward line in counter-clockwise direction.

To determine kinematical parameters of (r, θ) coordinates, we need to start from the problem's geometry. Referring to fig. 7.48, at this position,

$$x = R + R\cos 30 = 2.239 \text{ m}, \quad y = R - R\sin 30 = 0.6 \text{ m}$$

With these (x, y) intermediate coordinates, the polar coordinate description may be calculated with ease as

$$r = \sqrt{x^2 + y^2} = 2.318 \text{ m}, \quad \theta = \tan^{-1}\frac{y}{x} = 15^\circ$$

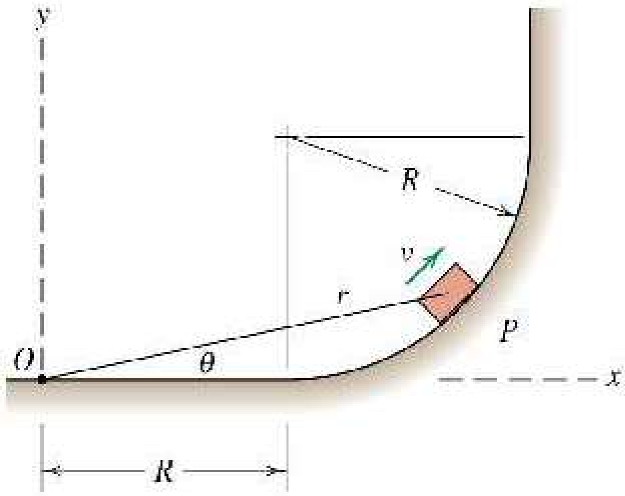


Figure 7.48: Example 7.25 ([4], pp. 78)

For the velocity, we may employ the n-t description due to the information given and the circular path with known geometry. At $\theta = 60°$, the constant velocity of 0.6 m/s has the direction as depicted in fig. 7.49. With the overlaying of the n-t and r-θ coordinate frames, we may project the velocity onto r- and θ-directions, and recall the formula to solve for the derivative parameters as follow.

$$[v_r = \dot{r}] \qquad v\cos 45 = \dot{r} = 0.424 \quad \text{m/s}$$

$$\left[v_\theta = r\dot{\theta}\right] \qquad v\sin 45 = 1.2 \times \dot{\theta}, \quad \dot{\theta} = 0.183 \quad \text{rad/s}$$

The acceleration of an object under circular motion with constant velocity is the centrifugal acceleration towards the center point. See fig. 7.49. Its magnitude is

$$\left[a_n = \frac{v^2}{\rho}\right] \qquad a_n = \frac{0.6^2}{1.2} = 0.3 = a$$

This may be decomposed into r- and θ-direction for which $\ddot{r}$ and $\ddot{\theta}$ may be resolved.

$$\left[a_r = \ddot{r} - r\dot{\theta}^2\right] \qquad -0.3\cos 45 = \ddot{r} - 1.2 \times 0.183^2, \quad \ddot{r} = -0.134 \quad \text{m/s}^2$$

$$\left[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}\right] \qquad 0.3\sin 45 = 1.2\ddot{\theta} + 2 \times 0.424 \times 0.183, \quad \ddot{\theta} = 0.0245 \quad \text{rad/s}^2$$

Example 7.26 ([4], Prob. 2/160) The slotted arm OA oscillates about O

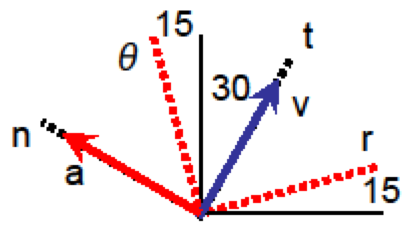


Figure 7.49: Solution to example 7.25

within the limits shown and drives the crank CP through the pin P. For an interval of the motion, $\dot{\theta} = K$, a constant. Determine the magnitude of the corresponding total acceleration of P for any value of θ within the range for which $\dot{\theta} = K$. Use polar coordinates r and θ. Show that the magnitudes of the velocity and acceleration of P in its circular path are constant.

Solution: The analysis will be start from the underlying geometry. The r- and θ-coordinates must always form the isosceles triangle OCP as drawn in fig. 7.51. Therefore $\beta = 2\theta$. It is given that the slotted arm OA rotates with the constant angular velocity $\dot{\theta} = K$. Consequently, the change of β may be determined as

$$\dot{\beta} = 2K, \quad \ddot{\beta} = 0$$

From the mechanism constraint, pin P moves along the circular path centering at C with the radius b and angle β. Therefore its velocity and acceleration may be determined as

$$v = \rho\dot{\beta} = 2bK = \text{constant}$$
$$a_n = v^2/\rho = 4bK^2, \quad a_t = \dot{v} = 0$$
$$a = 4bK^2 = \text{constant}$$

Example 7.27 ([3], Prob. 2/169) The earth satellite has a velocity $v = 17,970$ km/h as it passes the end of the semiminor axis at A. Gravitational attraction produces an acceleration $a = a_r = -1.556\text{m/s}^2$ as calculated from the gravitational law. For this position, calculate the rate $\dot{v}$ at which the speed of the satellite is changing and the quantity $\ddot{r}$.

Solution: The velocity must be tangent to the trajectory. With the satellite position at one end of the vertical semiminor axis, the velocity is in horizontal direction. Also, the n-t coordinate frame may be set up as depicted in fig. 7.53. Because the only acceleration of the satellite comes from the gravitational attraction, its direction points toward the earth along the radial line.

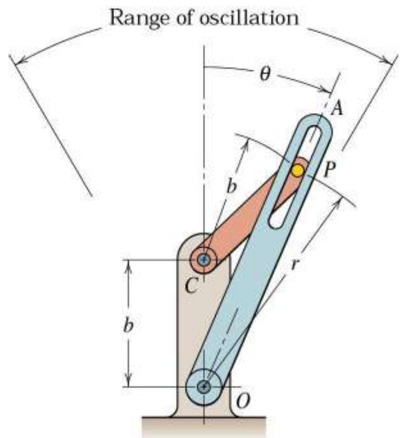


Figure 7.50: Example 7.26 ([4], pp. 79)

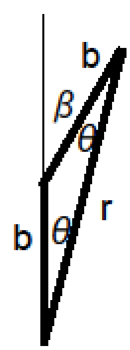


Figure 7.51: Solution to example 7.26

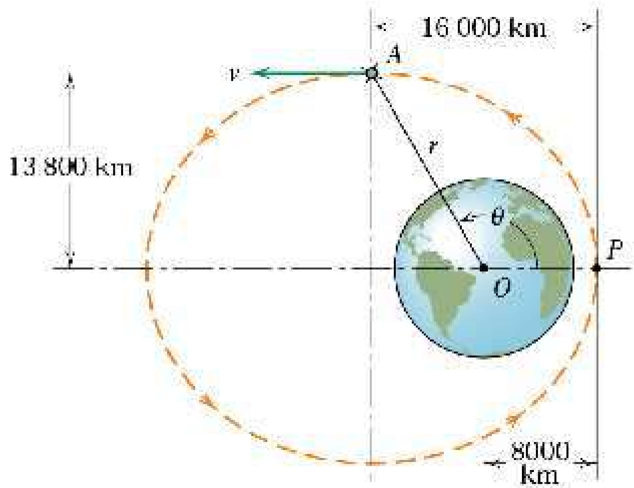


Figure 7.52: Example 7.27 ([3], pp. 80)

As shown in fig. 7.53, we may expressed both the velocity and the acceleration using r-θ coordinate system. Therefore, the rate $\dot{v}$ may be determined from the tangential acceleration:

$$a_t = -a\cos 60 = \dot{v} = -0.778 \text{ m/s}^2$$

One may conjecture that the rate $\ddot{r}$ may be determined using the expression of the radial acceleration. However, the rate $\dot{\theta}$ in the formula is not known yet. It may be found from the velocity relationship:

$$v_\theta = v\cos 30 = r\dot{\theta}, \quad \dot{\theta} = 2.7 \times 10^{-4} \text{ rad/s}$$

Substituting its value into the following relation, $\ddot{r}$ may now be solved.

$$a_r = a = -1.556 = \ddot{r} - r\dot{\theta}^2, \quad \ddot{r} = -0.388 \text{ m/s}^2$$

Example 7.28 ([4], Prob. 2/163) Pin A moves in a circle of 90 mm radius as crank AC revolves at the constant rate $\dot{\beta} = 60$ rad/s. The slotted link rotates about point O as the rod attached to A moves in and out of the slot. For the position $\beta = 30°$, determine $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$.

Solution: Motion of point A may be determined naturally using n-t coordinate system. Alternatively, the r-θ coordinate system may be used. At the current posture, two frames are oriented relative to each other as shown in fig. 7.55. Accordingly, the velocity lies along the t-axis and, due to the constant

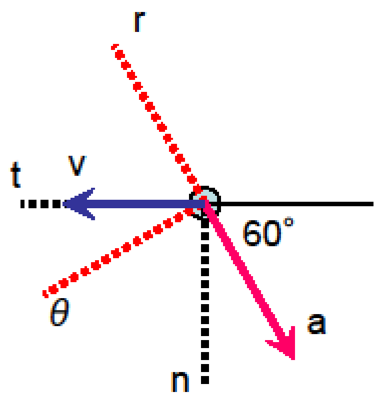


Figure 7.53: Solution to example 7.27

rotating rate of the crank AC, the acceleration points along the normal axis toward the center of rotation C.

Since the velocity and acceleration analysis calls for the position information as well, we will first employ the geometric constraint that the piston-crank mechanism forms the triangular loop OAC. Using the cosine law, we have

$$r^2 = 300^2 + 90^2 - 2 \times 300 \times 90 \cos 30, \quad r = 226.57 \text{ mm}$$

The angle θ may be determined from the sine law:

$$\frac{90}{\sin\theta} = \frac{r}{\sin 30}, \quad \theta = 11.46^\circ$$

The velocity and acceleration of A may be determined from the given information in n-t system as

$$v = \rho\dot{\beta} = 0.09 \times 60 = 5.4 \text{ m/s}$$

$$a = a_n = v^2/\rho = 5.4^2/0.09 = 324 \text{ m/s}^2$$

Decompose these vectors into r- and θ-components and match them with the expression in r-θ coordinates, we may solve for the derivatives of the parameters.

$$v_r = v\cos 48.54 = \dot{r} = 3.575 \text{ m/s}$$

$$v_\theta = v\sin 48.54 = r\dot{\theta}, \quad \dot{\theta} = 17.86 \text{ rad/s}$$

$$a_r = a\cos 41.46 = \ddot{r} - r\dot{\theta}^2, \quad \ddot{r} = 315 \text{ m/s}^2$$

$$a_\theta = -a\sin 41.46 = r\ddot{\theta} + 2\dot{r}\dot{\theta}, \quad \ddot{\theta} = -1510 \text{ rad/s}^2$$

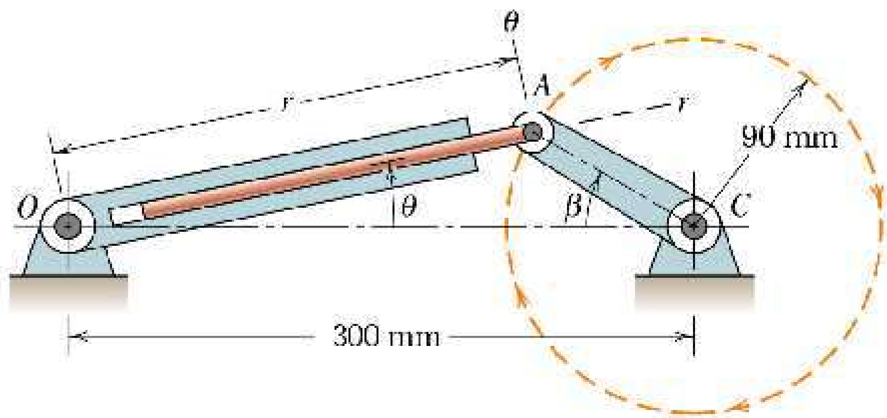


Figure 7.54: Example 7.28 ([4], pp. 80)

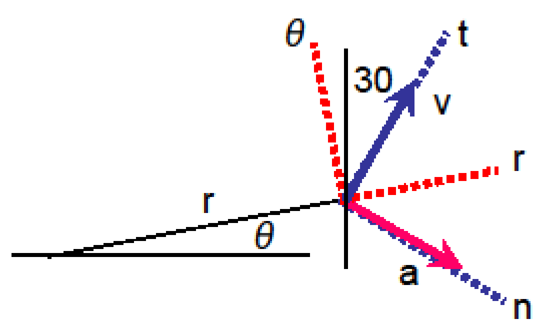


Figure 7.55: Solution to example 7.28

Example 7.29 ([4], Prob. 2/166) If the slotted arm is revolving CCW at the constant rate of 40 rev/min and the cam is revolving clockwise at the constant rate of 30 rev/min, determine the magnitude of the acceleration of the center of the roller A when the cam and arm are in the relative position for which $\theta = 30°$. The limacon has the dimensions $b = 100$ mm and $c = 75$ mm.

Solution: Polar equation of the cam profile (limacon type), $r = b - c\cos\theta$ is given. Therefore the appropriate coordinate system for this problem should be r-θ. However the angle in the coordinate must be measured with respect to a fixed reference direction. In other words, it must be the absolute angle. Since the cam is not fixed, the angle used in the equation must be compensated for this relative motion.

Let the angle θ be the positive CCW angle the slotted arm made with the horizontal reference line and β be the positive CW angle the concave of the cam made with the same horizontal reference line. Then, the limacon equation becomes

$$r = b - c\cos(\theta + \beta)$$

which, by the physical installation of the mechanism, indicates the radial distance of the roller A.

The angular velocity of the slotted arm and the cam are given to be constant of 40 and 30 rev/min, respectively. Hence, when the arm rotates to the position that $\theta = 30°$, the corresponding angle that the cam travels may be determined from the fact that the time spent are equal. For the slotted arm, the time spent is

$$\left[\omega = \tfrac{d\theta}{dt}\right] \qquad \int_0^{\pi/6} d\theta = \int_0^t \dot{\theta} dt, \quad t = 0.125 \text{ s}$$

For the cam, the angle rotated is

$$\int_0^{\beta} d\beta = \int_0^{0.125} \dot{\beta} dt, \quad \beta = 0.393 \text{ rad} = 22.5°$$

Therefore, the current radial distance is

$$r = 0.1 - 0.075\cos(30 + 22.5) = 54.3 \text{ mm}$$

Since the second derivative of θ and β are zero, the derivatives of r may be straightforwardly determined as

$$\dot{r} = c\left(\dot{\theta} + \dot{\beta}\right)\sin(\theta + \beta) = 0.436$$

$$\ddot{r} = c\left(\dot{\theta} + \dot{\beta}\right)^2\cos(\theta + \beta) = 2.453$$

Note that the angular coordinate to describe the roller is θ, *not* $\theta + \beta$. Now the acceleration may be evaluated:

$$a_r = \ddot{r} - r\dot{\theta}^2 = 2.453 - 0.0543\,(40 \times 2\pi/60)^2 = 1.5 \ \text{ m/s}^2$$

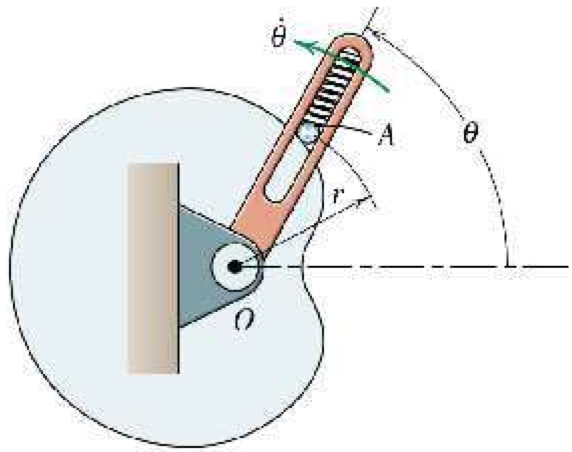


Figure 7.56: Example 7.29 ([4], pp. 80)

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 2 \times 0.436 \times (40 \times 2\pi/60) = 3.653 \ \ \text{m/s}^2$$
$$a = \sqrt{a_r^2 + a_\theta^2} = 3.95 \ \ \text{m/s}^2$$

7.7 Relative Motion (Translating Axes)

Motion of the objects in general are compound, i.e. they are both translating and rotating at the same time. Nevertheless, these complicated motion can be made easy by observing them with respect to the moving reference frames. For example, the absolute motion of the piston inside the engine block of a car going along the curvy road is difficult to describe. However, we will see the piston simply moves up and down along the cylinder had the observer been fixed with the engine block.

The motion seen is the *relative motion*. The *absolute motion* may be determined by combining the observed relative motion with the absolute motion of the moving reference frame. Hence, the strategy used is actually the 'divide and conquer'. The convolved absolute motion has been broken up into the relative motion and the easier absolute motion. As a consequence, it is quite crucial to select the appropriate moving frame for a specific problem.

We need to specify the observer frame first. The observer frame may, in general, be moving in both the translation and the rotation manner. However, as for an introduction to the relative motion, we will constrain the analysis to the one in *plane motion* which use the *pure translation moving reference frame*. This simply means that the relative motion will be observed on the moving reference frame that has no rotation.

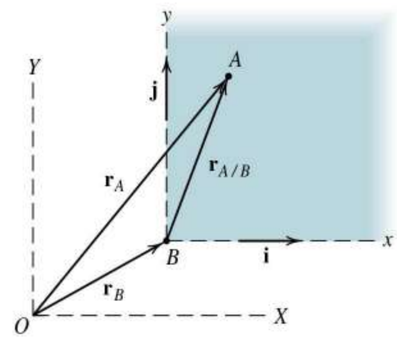


Figure 7.57: Absolute and relative motion of A with respect to B using the pure translating moving coordinate frame $\{xy\}$ ([3], pp. 91)

Consider fig. 7.57. Motion of A is observed from B where the measurements made are referenced with the pure translating moving coordinate frame $\{xy\}$. Its relative position of A with respect to B is denoted by the vector $\overline{r}_{A/B}$. We would like to determine the absolute position referenced with the fixed reference frame $\{XY\}$. Note that the axes of $\{XY\}$ and $\{xy\}$ is not necessary to be parallel to each other.

Let the absolute position vector of A and B (the origin of the observer frame) be $\overline{r}_A$ and $\overline{r}_B$ respectively. From the figure, we can set up the vector equation as

$$\overline{r}_A = \overline{r}_B + \overline{r}_{A/B} \tag{7.28}$$

which is frame independent. The separation between the vector and its description (which ties with the chosen frame) must be kept in mind. However, one must ensure the consistency of the description for all vector terms in a vector equation. If the vectors are originally represented in different frames, transformation between the two frames are required to make the equation be consistent. For $\overline{r}_{A/B}$, it is commonly described in the moving frame $\{xy\}$, which may be written explicitly as

$$\overline{r}_{A/B} = x\mathbf{i} + y\mathbf{j} \tag{7.29}$$

Differentiating eq. 7.28 to obtain the relative velocity equation:

$$\overline{v}_A = \overline{v}_B + \overline{v}_{A/B} \tag{7.30}$$

$\overline{v}_A$ and $\overline{v}_B$ are the absolute velocity of A and B while $\overline{v}_{A/B}$ is the relative velocity of A with respect to B. Its explicit expression is

$$\overline{v}_{A/B} = \dot{x}\mathbf{i} + x\dot{\mathbf{i}} + \dot{y}\mathbf{j} + y\dot{\mathbf{j}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \tag{7.31}$$

by denoting that $\{xy\}$'s motion is pure translation and so $\mathbf{i}$'s and $\mathbf{j}$'s direction do not change.

Similar arguments, that shall be skipped, may be used to derive for the relative acceleration equation:

$$\overline{a}_A = \overline{a}_B + \overline{a}_{A/B} \tag{7.32}$$

and

$$\overline{a}_{A/B} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} \tag{7.33}$$

It is trivial to show for the relative motion with the pure translating coordinate frame that

$$\overline{r}_{B/A} = -\overline{r}_{A/B}, \quad \overline{v}_{B/A} = -\overline{v}_{A/B}, \quad \overline{a}_{B/A} = -\overline{a}_{A/B} \tag{7.34}$$

Lastly, if the pure translating moving frame has constant velocity, $\overline{a}_B = \overline{0}$. From the relative acceleration equation, we can conclude that $\overline{a}_A = \overline{a}_{A/B}$. Implication of this result is that the determination of the absolute acceleration can also be made on the *inertial frame*, a pure translating frame that has no acceleration. Consequently, Newton's 2^{nd} law of motion holds in the *inertial* as well as in the *fixed* reference frame.

Example 7.30 ([3], Prob. 2/195) The car A has a forward speed of 18 km/h and is accelerating at 3m/s^2. Determine the velocity and acceleration of the car relative to observer B, who rides in a nonrotating chair on the ferris wheel. The angular rate $\omega = 3$ rev/min of the ferris wheel is constant.

Solution: Because the observer B rides in a nonrotating chair, car A is observed through the pure translating moving coordinate frame. Velocity and acceleration of the car are given to be

$$\overline{v}_A = 18 \times \frac{10}{36}\mathbf{i} = 5\mathbf{i} \;\; \text{m/s}$$

and

$$\overline{a}_A = 3\mathbf{i} \;\; \text{m/s}^2$$

using the x-y coordinate frame depicted in fig. 7.58.

The observer B is moving in circular motion. Hence his velocity and acceleration may be calculated from the specified angular velocity and acceleration of the ferris wheel:

$$\overline{v}_B = \rho\dot{\beta}\,(\cos 45\mathbf{i} - \sin 45\mathbf{j}) = 2\mathbf{i} - 2\mathbf{j} \;\; \text{m/s}$$

and

$$\overline{a}_B = \rho\dot{\beta}^2\,(-\cos 45\mathbf{i} - \sin 45\mathbf{j}) = -0.628\mathbf{i} - 0.628\mathbf{j} \;\; \text{m/s}^2$$

on the fact that the tangential component is null due to constant angular velocity. Consequently, the velocity and acceleration of A relative to B may be determined directly as

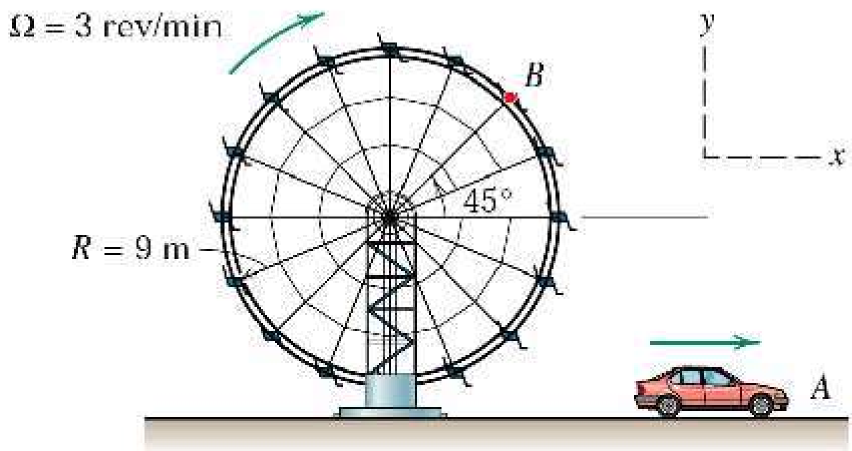


Figure 7.58: Example 7.30 ([3], pp. 96)

$[\overline{v}_{A/B} = \overline{v}_A - \overline{v}_B]$ $\qquad \overline{v}_{A/B} = 5\mathbf{i} - (2\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} + 2\mathbf{j}$ m/s

$[\overline{a}_{A/B} = \overline{a}_A - \overline{a}_B]$ $\qquad \overline{a}_{A/B} = 3\mathbf{i} - (-0.628\mathbf{i} - 0.628\mathbf{j}) = 3.628\mathbf{i} + 0.628\mathbf{j}$ m/s^2

Example 7.31 ([3], Prob. 2/197) Hockey player A carries the puck on his stick and moves in the direction shown with a speed $v_A = 4$ m/s. In passing the puck to his stationary teammate B, by what shot angle α should the direction of his shot trail the line of sight if he launches the puck with a speed of 7 m/s relative to himself?

Solution: The puck must be passed from the player A to B with the direction being 45° relative to his movement. Because the player A is moving, he will see the puck moving in the direction of $(45 + \alpha)^\circ$ instead. See the relevant velocity diagram corresponding to the relative velocity relationship

$$\overline{v}_P = \overline{v}_A + \overline{v}_{P/A}$$

in fig. 7.60. Using the law of sine with this triangle, the shot angle α may be determined:

$$\frac{7}{\sin 45} = \frac{4}{\sin \alpha}, \quad \alpha = 23.8^\circ$$

Example 7.32 ([3], Prob. 2/210) The aircraft A with radar detection equipment is flying horizontally at 12 km and is increasing its speed at the rate of 1.2 m/s each second. Its radar locks onto an aircraft flying in the same direction and in the same vertical plane at an altitude of 18 km. If A has a speed of 1000

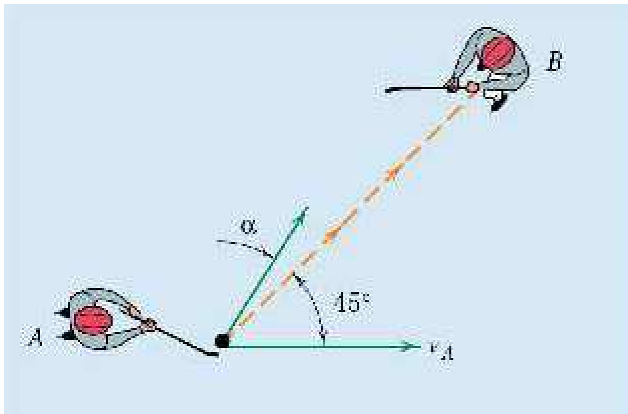


Figure 7.59: Example 7.31 ([3], pp. 97)

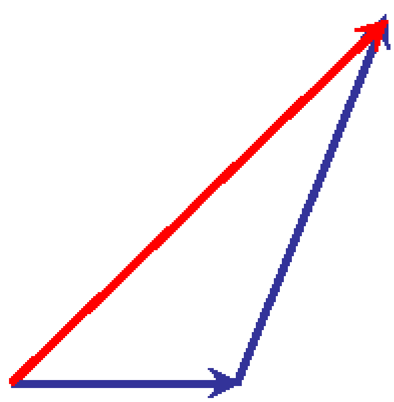


Figure 7.60: Solution to example 7.31

km/h at the instant that $\theta = 30°$, determine the values of $\ddot{r}$ and $\ddot{\theta}$ at this same instant if B has a constant speed of 1500 km/h.

Solution: Since the velocity and acceleration of A and B are provided, their relative motion may be determined. If we define the x-axis to be the horizontal axis pointing to the right,

$$\left[\overline{v}_{B/A} = \overline{v}_B - \overline{v}_A\right] \qquad \overline{v}_{B/A} = 1500 \times \tfrac{10}{36}\mathbf{i} - 1000 \times \tfrac{10}{36}\mathbf{i} = 138.89\mathbf{i} \ \ \text{m/s}$$

$$\left[\overline{a}_{B/A} = \overline{a}_B - \overline{a}_A\right] \qquad \overline{a}_{B/A} = \overline{0} - 1.2\mathbf{i} = -1.2\mathbf{i} \ \ \text{m/s}^2$$

These relative vectors may be equivalently represented in other coordinate frames, for which we choose the r-θ coordinate frame due to the required parameters. The frame is conventionally affixed to B, depicted in fig. 7.61. We may then decompose these vectors geometrically and equate them to the r-θ parameters formulation. Hence the values of $\ddot{r}$ and $\ddot{\theta}$ may be found.

$$\left[\overline{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\right] \qquad 138.89\cos 30\mathbf{e}_r - 138.89\sin 30\mathbf{e}_\theta = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\dot{r} = 120.28 \ \ \text{m/s}, \quad \dot{\theta} = -5.787 \times 10^{-3} \ \ \text{rad/s}$$

$$\left[\overline{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta\right]$$

$$-1.2\cos 30\mathbf{e}_r + 1.2\cos 60\mathbf{e}_\theta = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

$$\ddot{r} = -0.637 \ \ \text{m/s}^2, \quad \ddot{\theta} = 1.66 \times 10^{-4} \ \ \text{rad/s}^2$$

Example 7.33 ([3], Prob. 2/211) A batter hits the baseball A with an initial velocity of $v_o = 30$ m/s directly toward fielder B at an angle of 30° to the horizontal; the initial position of the ball is 0.9 m above the ground level. Fielder B requires $\frac{1}{4}$ sec to judge where the ball should be caught and begins moving to that position with constant speed. Because of great experience, fielder B choose his running speed so that he arrives at the "catch position" simultaneously with the baseball. The catch position is the field location at which the ball altitude is 2.1 m. Determine the velocity of the ball relative to the fielder at the instant the catch is made.

Solution: The ball was hit and then moved freely in the air subject to the constant gravitational acceleration

$$\overline{a} = -g\mathbf{j}$$

This implies the zero acceleration component in the horizontal direction and

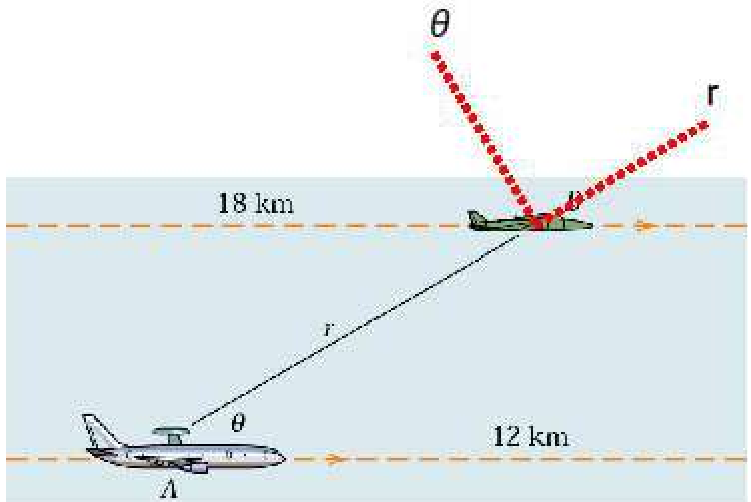


Figure 7.61: Example 7.32 ([3], pp. 100)

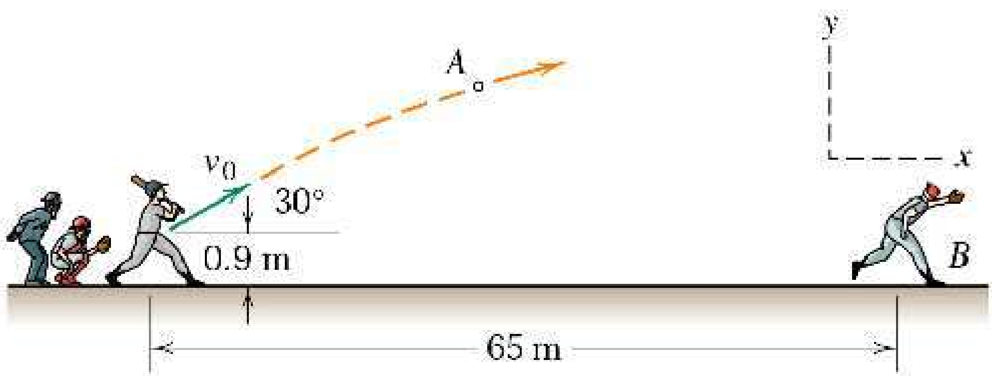


Figure 7.62: Example 7.33 ([3], pp. 100)

hence its associated velocity is constant

$$\overline{v}_x = 30\cos 30\mathbf{i} = 25.98\mathbf{i} \;\; \text{m/s}$$

Vertical component of the velocity is changing with constant acceleration. Hence the value at the catch position becomes

$$[v^2 = v_o^2 + 2a(s - s_o)] \qquad v_y^2 = (30\sin 30)^2 - 2g\,(2.1 - 0.9)$$

$$\overline{v}_y = -14.19\mathbf{j} \;\; \text{m/s}$$

Therefore the velocity of the ball at the catch position is

$$\overline{v}_A = 25.98\mathbf{i} - 14.19\mathbf{j} \;\; \text{m/s}$$

To determine its relative velocity, the catcher's velocity is necessary. This may be determined using the fact that the time spent for both the ball and the catcher is the same. Integrating the vertical component of the ball velocity during the projectile motion, we obtain the relationship between the displacement and the time spent:

$$\left[s = s_o + v_o t + \tfrac{1}{2}at^2\right] \qquad 2.1 = 0.9 + (30\sin 30)\,t - \tfrac{1}{2}gt^2$$

$$t = 2.976 \text{ s}$$

With this information, the horizontal displacement of the ball is

$$[s = vt] \qquad s_x = 30\cos 30 \times 2.976 = 77.32 \text{ m} \;\; \text{to the right of the batter}$$

This implies, from fig. 7.62, that the catcher B must move by the distance $77.32 - 65 = 12.32$ m to the right, to meet the ball, in $2.976 - 0.25 = 2.726$ second; due to the idling thinking time. Hence the constant running velocity of the catcher is

$$[v = s/t] \qquad \overline{v}_B = \tfrac{12.32}{2.726} = 4.52\ \mathbf{i} \;\; \text{m/s}$$

Consequently, the velocity of the ball relative to the catcher at the time of catching is

$$\left[\overline{v}_{A/B} = \overline{v}_A - \overline{v}_B\right] \qquad \overline{v}_{A/B} = (25.98\mathbf{i} - 14.19\mathbf{j}) - 4.52\mathbf{i} = 21.46\mathbf{i} - 14.19\mathbf{j} \;\; \text{m/s}$$

Chapter 8

Kinetics of Particles

8.1 Introduction

This chapter is a continuation of the previous one where now the kinetics of particles will be investigated. Kinetics is the study of the relations between the forces and the motion. As for the introductory material, we will not seriously concern whether the forces cause the motion or the motion generates the forces. This is the issue called causality problem.

The object of interested in this chapter will be confined to the particles. Relatively speaking, the body whose physical dimensions are so small compared with the radius of curvature of its trajectory may safely be treated as the particle.

There are at least three approaches for solving the kinetics problems. They are (a) Newton-Euler's method (b) work and energy method and (c) momentum method. Each has its own pros and cons. Nevertheless, we will be employing only Newton-Euler's method. Roughly, Newton's law governs the force and translational motion while Euler's law relates the moment to the rotational motion. Since the concept of the rotation does not apply for the particle object, only Newton's second law of motion will be mentioned in this chapter.

8.2 Newton's Second Law

Newton's second law of motion states:

> "The *absolute* acceleration of a *particle* is proportional to the resultant force acting on it and is in the direction of this resultant force."

This statement can be formulated as

$$\Sigma \mathbf{F} = m\mathbf{a} \tag{8.1}$$

where m = mass (resistance to rate of change of velocity) of the *particle*
$\mathbf{F}$ = *resultant force* acting on the particle
$\mathbf{a}$ = resulting acceleration measured in a *nonaccelerating frame* of reference

For most engineering problems on earth, the acceleration measured with respect to the reference frame fixed to the earth's surface may be treated as absolute. Hence it may be used in Newton's law. Sometimes, the absolute acceleration may be determined via the use of relative motion. If the pure translating moving reference frame is used, the (absolute) acceleration of A might be calculated from

$$\overline{a}_A = \overline{a}_B + \overline{a}_{A/B}$$

Another caveat about this empirical law is that it does not hold when the velocity of the order of the speed of light is involved. In that case, reader is urged to refer to advanced topic on theory of relativity.

8.3 Equations of Motion

Newton's second law of motion is typically used to formulate the equation of motion. Because the equation is of vector type, one first need to decide which coordinate system will be employed. Then, the vectorial equation can be decomposed into scalar equations accordingly. Since the equation involves the acceleration, however, it is common to perform kinematic analysis prior to embark on Newton's law.

In the following, some common issues in applying Newton's law will be mentioned.

Two problems of dynamics

The problems involving using Newton's second law may be classified into two types called inverse and forward dynamics.

1. *Inverse dynamic* In this case, the kinematic conditions, such as coordinate parameters and their derivatives, are specified. In turn, the corresponding force will be determined. This problem is the straightforward application of Newton's law because the force term is explicitly separated. Regarding to the unknown force, the equations are just simple algebraic equations.

2. *Forward dynamic* Conversely, the applied force are provided. Rather, the resulting motion will be determined, whether at the instant or as a function of time.It is far more difficult to solve compared to the inverse dynamic problems, because the type of equations are a system of differential equations. For simple form of forcing function, we may be able to determine the closed form solution, as for the rectilinear motion problems. Unfortunately, the force may be described as some function of mixing parameters of time, displacement, velocity, and acceleration. Then, only the numerical solution may be obtained.

Unconstrained motion

As the name suggested, unconstrained motion is the motion free of constraints. Behavior of the motion is determined by the initial conditions and the explicit external forces. A well known example would be the projectile motion. For particle under free motion in general has three degrees of freedom. Therefore three scalar equations of motion may be applied and integrated to obtain the motion.

Constrained motion

In this case, motion of the particle is partially or totally constrained (and determined) by restraining guides, for example. Its initial motion and explicit external forces still influence the motion, of course. Hence *all forces*, i.e. both *applied* and *reactive* forces (or external force and interaction force with the constraints), that act *on* the particle must be accounted for in Newton's law of motion. The number of degrees of freedom and associated differential equations of motion may be reduced, depending on the type of constraints.

Free body diagram
In formulating the equations of motion, *all forces* acting *on* the particle need to be accounted. Free body diagram, introduced in chapter 3, is a systematic graphical method that unveils every force acting on the isolated body. Only after the free body diagram has been completed should the equations of motion be developed. The appropriate coordinate systems should be selected and consistently employed throughout the problem. System involving with particles only is particularly simple because there is no concept about dimension of the particle. Implicitly, the relevant forces may be treated as concurrent acting through the center of mass.

8.4 Rectilinear Motion

We will start from the simple case where the particle is constrained to move along the straight line, i.e. the rectilinear motion. If the x-axis lies along with that direction, we may apply Newton's law component-wise as

$$\Sigma F_x = ma_x \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0$$

Unfortunately, sometimes we are not free to assign a coordinate axis along the motion direction. In that case the nonzero acceleration component will be shown up in all equations. The application of Newton's law would become

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z$$

Additionally, in several cases, other coordinate systems, such as n-t or r-θ, might be more appropriate.

Example 8.1 ([3], Prob. 3/19) The coefficient of static friction between the flat bed of the truck and the crate it carries is 0.30. Determine the minimum stopping distance s that the truck can have from a speed of 70 km/h with constant deceleration if the crate is not to slip forward.

Solution: If the crate is not to slip forward, motion of it and the truck must be the same at all time. This implies they must have the same acceleration. Additionally, it is required that the truck decelerates with a maximum constant deceleration that the crate still not slipping forward. This induces the friction force between the truck bed and the crate to reach the maximum value. That is the crate is on the impending status.

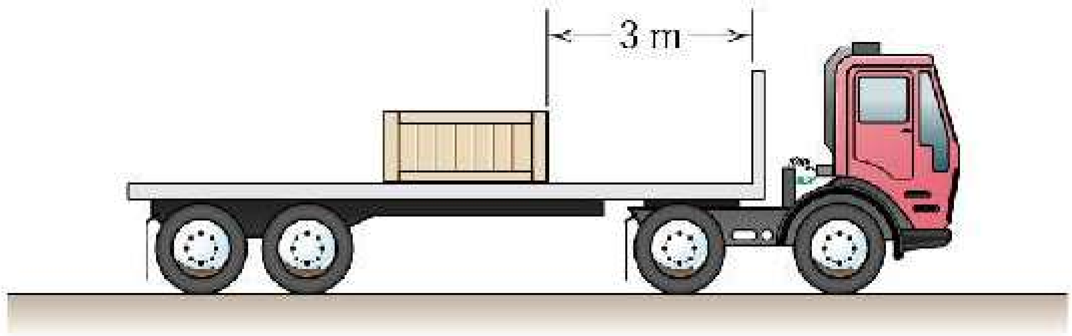


Figure 8.1: Example 8.1 ([3], pp. 134)

To determine the deceleration, we draw the free body diagram of the crate and assign the x-axis, as shown in fig. 8.2. By the Coulomb's model of friction, and the specified coefficient of static friction, μ_s, of 0.3, the static friction force acting on the crate is

$$[F_s = \mu_s N] \qquad F_s = 0.3 \times mg$$

Applying Newton's law along the x-direction, the unknown deceleration is determined.

$$[\Sigma F_x = ma_x] \qquad -0.3mg = ma_x, \quad a_x = -0.3g$$

which must be kept constant for the condition of minimum stopping distance s. This is also the deceleration of the truck, from which we may determine its traveling distance from the given velocity (70 km/h) before stopping.

$$[v^2 = v_o^2 + 2a(s - s_o)] \qquad 0 = \left(70 \times \tfrac{10}{36}\right)^2 + 2\,(-0.3g)\,s, \quad s = 64.2 \text{ m}$$

Example 8.2 ([3], Prob. 3/20) If the truck of previous problem comes to stop from an initial forward speed of 70 km/h in a distance of 50 m with uniform deceleration, determine whether or not the crate strikes the wall at the forward end of the flat bed. If the crate does strike the wall, calculate its speed relative to the truck as the impact occurs. Use the friction coefficients $\mu_s = 0.3$ and $\mu_k = 0.25$.

Solution: From the result of the previous problem, we see that the stopping distance of 50 m is less than the minimum value of 64.2 m for non-slipping condition. Therefore the crate will slip. As a result, the motion of the truck and the crate will not be the same. For the truck, its uniform deceleration is

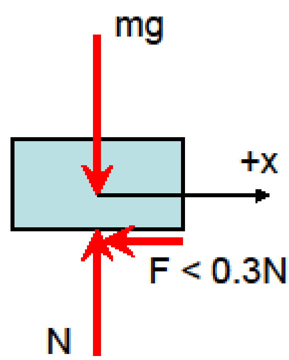


Figure 8.2: Solution to example 8.1

$$[v^2 = v_o^2 + 2a(s - s_o)] \quad 0 = \left(70 \times \tfrac{10}{36}\right)^2 + 2a_{\text{truck}} \times 50, \quad a_{\text{truck}} = -3.781 \ \text{ m/s}^2$$

which allows us to calculate the time used to stop:

$$[v = v_o + a(t - t_o)] \qquad 0 = \left(70 \times \tfrac{10}{36}\right) - 3.781 t_{\text{stop}}, \quad t_{\text{stop}} = 5.14 \text{ s}$$

Consider the crate. With the material properties of the mating surface, the static and kinetic friction force may be calculated as

$$[F_s = \mu_s N] \qquad F_s = 0.3mg = 2.943m$$

$$[F_k = \mu_k N] \qquad F_k = 0.25mg = 2.45m$$

Assume the crate and the truck go together as before, then $a_{\text{truck}} = a_{\text{crate}}$. From the free body diagram of the crate (fig. 8.4), applying Newton's law to determine the required friction force:

$$[\Sigma F_x = ma_x] \qquad -F = m\,(-3.781)$$

which is greater than the static friction. This indicates the assumed situation is not possible. The crate must then slip and the friction drops to kinetic friction. Re-applying Newton's law, the acceleration of the crate may then be determined.

$$[\Sigma F_x = ma_x] \qquad -2.45m = ma_{\text{crate}}, \quad a_{\text{crate}} = -2.45 \ \text{ m/s}^2$$

To determine whether the crate strikes the wall, the analysis must be based upon the relative motion because both the truck and the crate move. Their relative acceleration is

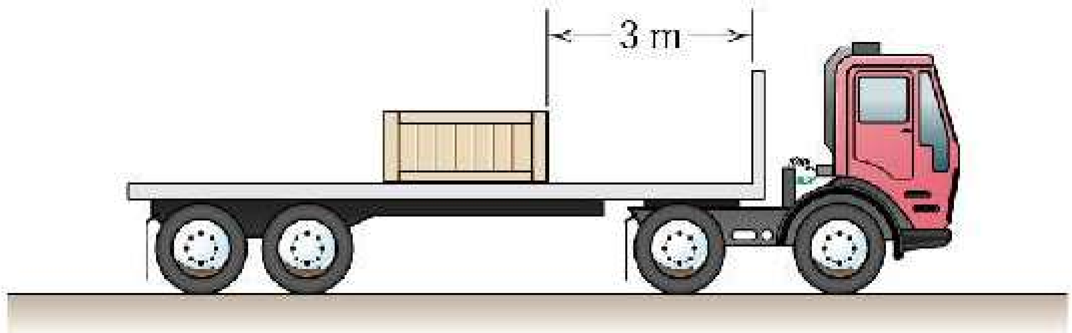


Figure 8.3: Example 8.2 ([3], pp. 134)

$\left[a_{\mathrm{c/t}} = a_{\mathrm{c}} - a_{\mathrm{t}}\right] \qquad a_{\mathrm{c/t}} = -2.45 - (-3.781) = 1.331 \;\; \mathrm{m/s^2}$

The crate is moving with forward acceleration relative to the truck. Hence it will slip forward. So we need to determine whether it will strike the wall.

To do so, we will calculate the time that the crate need for traveling 3 m relative to the truck bed. Since the relative acceleration is constant, relationship between the displacement and the time spent is simplified to

$$s = s_o + v_o(t - t_o) + \frac{1}{2}a(t^2 - t_o^2)$$

Note that for this problem, the kinematical quantities in the formula refer to their relative values. Therefore,

$$3 = \frac{1}{2} \times 1.331 \times t^2, \quad t_{\mathrm{strike}} = 2.123 \text{ s}$$

is the time spent for the crate to hit the wall, which is *less than* the time for the truck to come to stop. Consequently, the crate will strike the wall before the truck stops. The relative speed as the impact occurs may be calculated from

$[v = v_o + a(t - t_o)] \qquad v_{\mathrm{c/t}} = 0 + 1.331 \times 2.123 = 2.826 \text{ m/s}$

Example 8.3 ([3], Prob. 3/23) If the coefficients of static and kinetic friction between the 20-kg block A and the 100-kg cart B are both essentially the same value of 0.50, determine the acceleration of each part for (a) $P = 60$ N and (b) $P = 40$ N.

Solution: From our intuition, if we apply too strong pulling force, block A will move relatively to the right of block B. However if the applied force is small

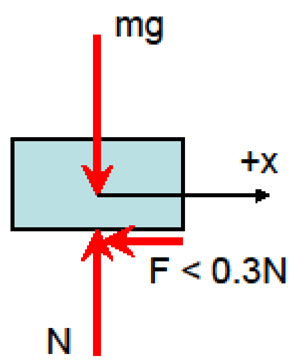


Figure 8.4: Solution to example 8.2

enough, both blocks will move together. In either case, the free body diagram will be as shown in fig. 8.6. The normal force between the blocks, N_A, is to balance the weight of block A. Therefore $N_A = 20g$ N. In turn, the maximum supplied friction force (static) would be

$$F_{\max} = 0.5N_A = 98.1 \text{ N}$$

(a) Due to the cable-pulley arrangement, the effective pulling force is 120 N which is greater than maximum resisting friction force $F_{\max}$. Hence block A will slip forward relative to B. The corresponding friction force is the kinetic frition, which happens to be equal to the static value for this problem. Their accelerations are governed by Newton's law:

$$[\Sigma F_x = ma_x] \qquad 120 - 98.1 = 20a_A, \quad a_A = 1.095 \ \text{ m/s}^2$$

$$98.1 = 100a_B, \quad a_B = 0.981 \ \text{ m/s}^2$$

(b) Now the effective pulling force is 80 N which is less than $F_{\max}$. Hence block A will not slip relative to B. Since they are moving together, we may treat them as a unified block of 120 kg subject to the pulling force of 80 N. Therefore, the acceleration becomes

$$[\Sigma F_x = ma_x] \qquad 80 = 120a, \quad a = 0.667 \ \text{ m/s}^2$$

Developing friction force may be determined from either the isolated free body diagram of block A or B. If we use block A, then

$$[\Sigma F_x = ma_x] \qquad 80 - F = 20 \times 0.667, \quad F = 66.67 \text{ N}$$

which is less than $F_{\max}$. Hence the assumption is valid.

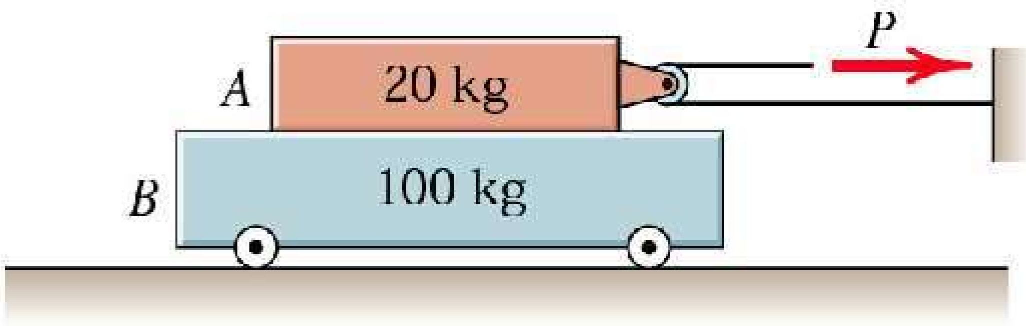


Figure 8.5: Example 8.3 ([3], pp. 134)

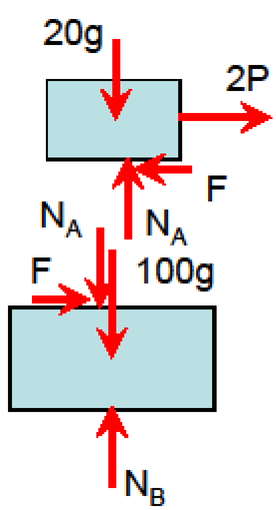


Figure 8.6: Solution to example 8.3

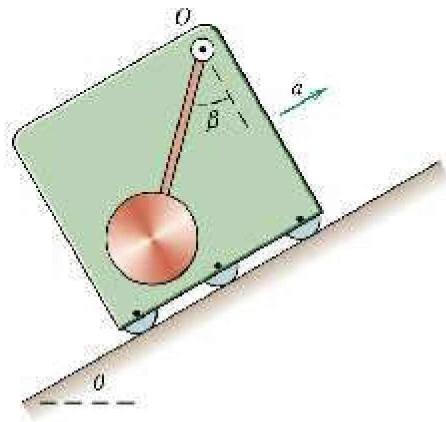


Figure 8.7: Example 8.4 ([4], pp. 132)

Example 8.4 ([4], Prob. 3/18) A simple pendulum is pivoted at O and is free to swing in the vertical plane of the plate. If the plate is given a constant acceleration a up the incline θ, write an expression for the steady angle β assumed by the pendulum after all initial start-up oscillations have ceased. Neglect the mass of the slender supporting rod.

Solution: When the pendulum reaches the steady state motion, it will possess the same acceleration as of the plate, namely a upward the incline. From the free body diagram in fig. 8.8, there are two forces, i.e. tension and gravity force, acting on the concentrated swinging mass, with their lines of action shown. Choosing the coordinate frame x-y aligning with its natural motion description, Newton's law may then be set up conveniently;

$$[\Sigma F_y = 0] \qquad T\cos\beta - mg\cos\theta = 0$$

$$[\Sigma F_x = ma_x] \qquad T\sin\beta - mg\sin\theta = ma$$

From these algebraic trancendental equations, the steady angle β may be determined.

$$\beta = \tan^{-1}\left(\frac{a + g\sin\theta}{g\cos\theta}\right)$$

Example 8.5 ([3], Prob. 3/26) For the friction coefficients $\mu_s = 0.25$ and $\mu_k = 0.20$, calculate the acceleration of each body and the tension T in the cable.

Solution: The constraint in this problem needs to be analyzed explicitly, or we will not be able to relate the motion of A and B. We observe that the

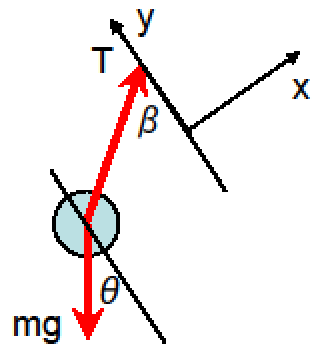


Figure 8.8: Solution to example 8.4

cable-pulley system constitutes the motion constraint, or the motion relationship, between A and B. The constraint stems from the underlying property of the cable that, no matter how the two blocks move, its length must be constant. Of course, we assume that the elasticity of the cable can be neglected.

To relate this fact with the block motion, we must express its length in terms of the block displacement. With additional assumptions that the cable is always taut and there is no slipping between the cable and the pulley's groove, we may then write the cable's length, l, as

$$s_A + 2s_B + c = l$$

where s_A and s_B are the displacement of A and B measured from the center of the pulley positively along the incline (for A) and downward (for B). The constant c is to take care for the portions that wrap around the pulleys, for example. Differentiating the constraint equation to come at its acceleration form;

$$a_A + 2a_B = 0$$

Hence the downward motion of A will be twice as fast as the upward motion of B, as one may understand.

For kinetic analysis, we start by drawing the free body diagram of both objects. See fig. 8.10. Since block A involves with the friction force, one may prepare for the calculation of the normal force and maximum supplied friction force:

$[\Sigma F_y = 0]$ $$N = 60g\cos 30 \text{ N}$$

$[F_{\max} = \mu_s N]$ $$F_{\max} = 0.25 \times 60g\cos 30 = 127.4 \text{ N}$$

We are not certain whether block A will move downward, so we assume it is not yet moving but impends. That is block A is in equilibrium with the friction force developing to the maximum value.

$$[\Sigma F_x = 0] \qquad 60g\sin 30 - F_{\max} - T = 0, \quad T = 166.9 \text{ N}$$

This tension force is transmitted to block B, in which we may apply Newton's law to it:

$$[\Sigma F_{y'} = ma_{y'}] \qquad 20g - 2T = 20a_{y'}, \quad a_{y'} = -6.88 \;\; \text{m/s}^2$$

indicating that block B is moving upward. This is in contradiction to the static equilibrium assumption and the kinematic constraint. It therefore implies that block A must be in motion. Now, should it be sliding upward or downward? A little thought will help clarify. If A is up, B must be down. This requires $2T$ to be smaller than $20g$. In turn, T will likely to be smaller than $60g\sin 30$. Hence A will be down, which is not agree to the proposition.

Therefore, block A must go downward and B upward. The friction force will become kinetic friction. Applying Newton's law to block A and B, and recognizing the kinematic constraint derived earlier, we have

$$[\Sigma F_x = ma_x] \qquad 60g\sin 30 - F_k - T = 60a_A$$

$$[\Sigma F_{y'} = ma_{y'}] \qquad 20g - 2T = 20a_B$$

$$a_A + 2a_B = 0$$

Solving these three equations simultaneously, we have

$$T = 105.35 \text{ N}, \quad a_B = -0.725 \;\; \text{m/s}^2, \quad a_A = 1.45 \;\; \text{m/s}^2$$

Example 8.6 ([3], Prob. 3/35) A bar of length l and negligible mass connects the cart of mass M and the particle of mass m. If the cart is subject to a constant acceleration a to the right, what is the resulting steady-state angle θ that the freely pivoting bar makes with the vertical? Determine the net force P (not shown) that must be applied to the cart to cause the specified acceleration.

Solution: Free body diagram of the cart and the pendulum are shown in fig. 8.12. Because the system is to be considered when the pendulum attains the steady angle, both travel with the same motion, i.e. the constant acceleration a to the right. The suitable coordinate system is then the x-y coordinate frame shown.

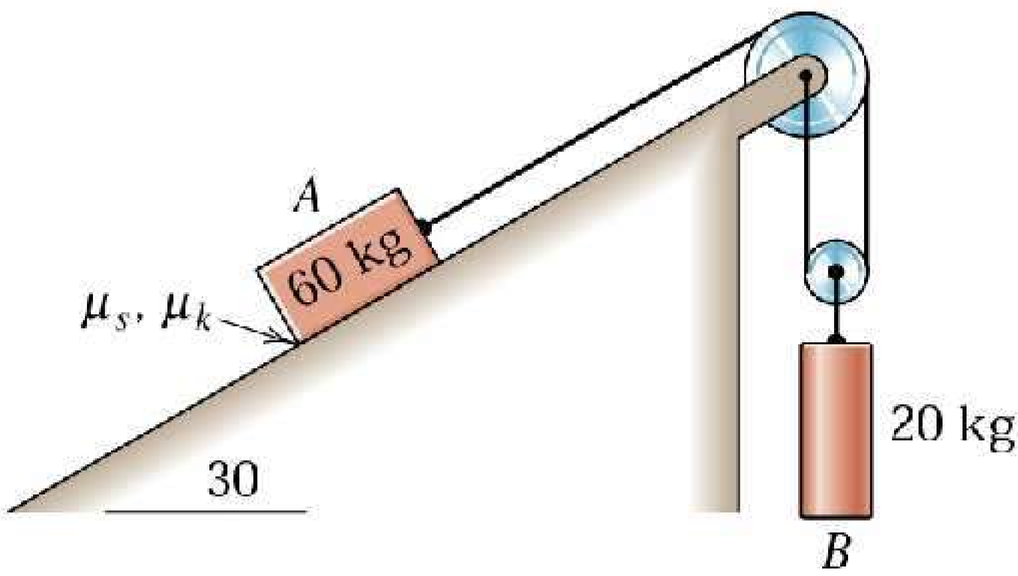


Figure 8.9: Example 8.5 ([3], pp. 135)

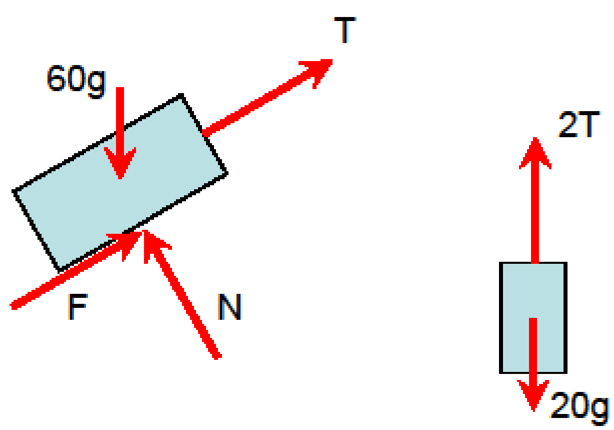


Figure 8.10: Solution to example 8.5

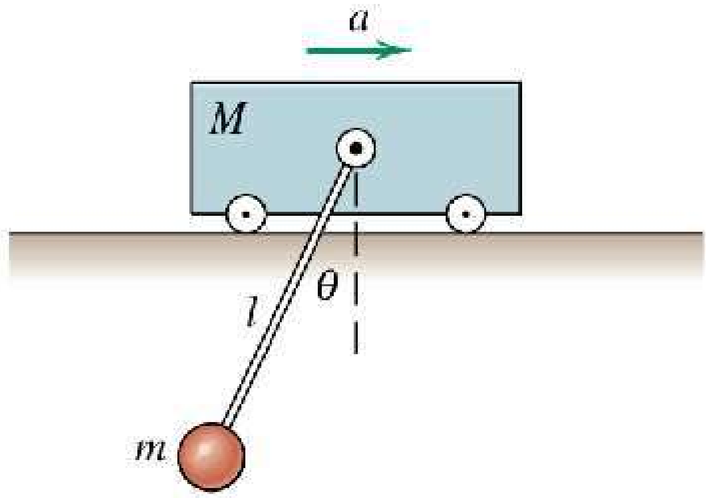


Figure 8.11: Example 8.6 ([3], pp. 137)

Applying the Newton's law to the pendulum:

$$[\Sigma F_y = 0] \qquad T\cos\theta - mg = 0, \quad T = \tfrac{mg}{\cos\theta}$$

$$[\Sigma F_x = ma_x] \qquad T\sin\theta = ma, \quad \theta = \tan^{-1}\left(\tfrac{a}{g}\right)$$

The applied force P may be determined from the equation of motion of the cart:

$$[\Sigma F_x = ma_x] \qquad P - T\sin\theta = Ma, \quad P = (m+M)\,a = (m+M)\,g\tan\theta$$

Example 8.7 ([3], Prob. 3/38) Determine the accelerations of bodies A and B and the tension in the cable due to the application of the 300 N force. Neglect all friction and the masses of the pulleys.

Solution: Free body diagram of A and B are drawn in fig. 8.14 where the tension in the cable is denoted T. Let the positive displacement of each body be in the direction measured from the referencing bump to the body. See fig. 8.14. Applying Newton's law to each of them in succession, we have

$$[\Sigma F_x = ma_x] \qquad -2T = 70a_A$$

$$300 - 3T = 35a_B$$

There are three unknowns, namely a_A, a_B, and T. Hence another equation is needed. It will be formed by the kinematic constraint. A and B cannot move

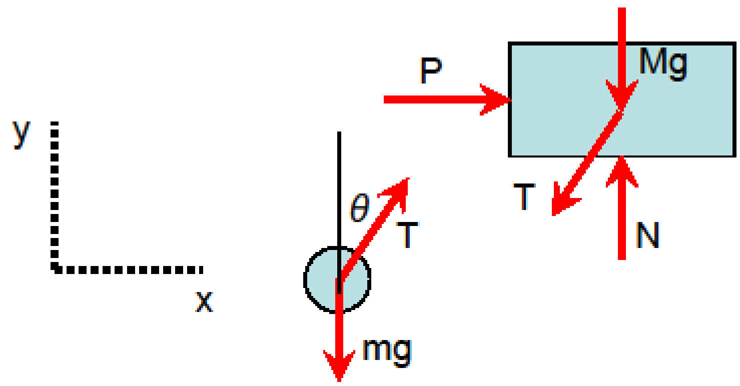


Figure 8.12: Solution to example 8.6

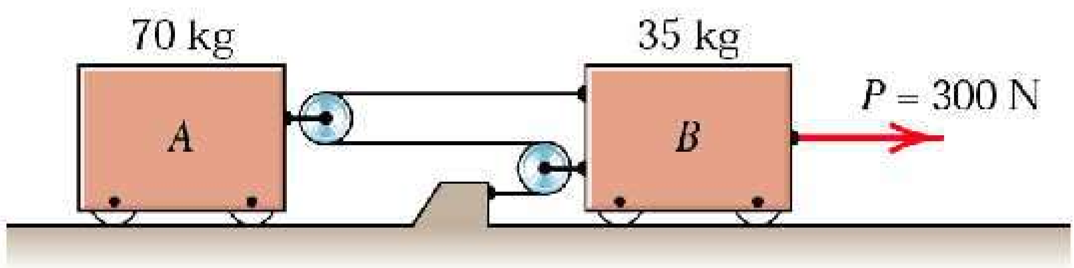


Figure 8.13: Example 8.7 ([3], pp. 137)

independently, but are related by the cable-pulley mechanism. With the ideal assumption of inelasticity and rolling without slipping, the cable length, l, may be expressed as

$$2s_A + 3s_B + c = l$$

where c takes care of all constant lengths along the cable route. Differentiating the relation, we obtain the acceleration relationship

$$2a_A + 3a_B = 0$$

that shall be used in solving for the unknowns. The answers would be

$$a_A = -2.34 \ \ \mathrm{m/s^2}, \ \ a_B = 1.56 \ \ \mathrm{m/s^2}, \ \ T = 81.8 \ \mathrm{N}$$

Example 8.8 ([3], Prob. 3/45) The sliders A and B are connected by a light rigid bar and move with negligible friction in the slots, both of which lie in a horizontal plane. For the position shown, the velocity of A is 0.4 m/s to the

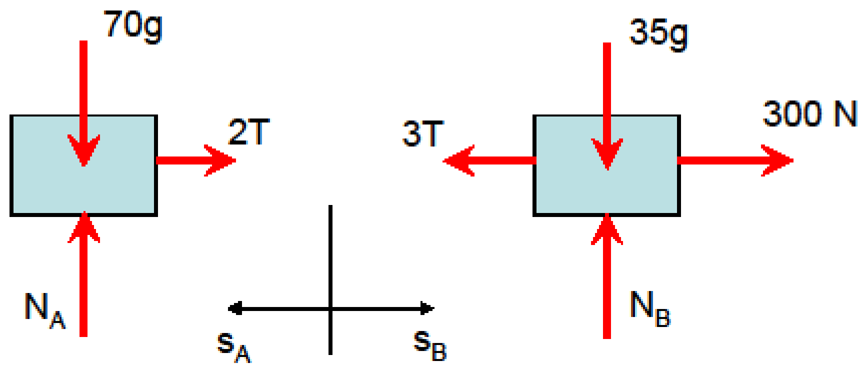


Figure 8.14: Solution to example 8.7

right. Determine the acceleration of each slider and the force in the bar at this instant.

Solution: Kinematic constraint of this problem is a little bit involved due to the nonlinearity of the geometry. It can be formulated by observing that motion of A and B are constrained by the 0.5 m bar. Altogether, they form the shape-changing triangle OAB. Let the displacement of A and B, s_A and s_B, be measured positively from the fixed apex O along the slots as shown in fig. 8.16. For this particular instant,

$$s_A = s_B \quad \text{and} \quad 0.5 = s_A \cos 15 + s_B \cos 15$$

$$s_A = s_B = 0.2588 \text{ m}$$

Change in s_A and s_B are governed by the cosine law as

$$l^2 = s_A^2 + s_B^2 - 2 s_A s_B \cos 150$$

Differentiating the equation to obtain the velocity relationship, we have

$$0 = 2 s_A v_A + 2 s_B v_B - 2 \cos 150 \left(s_A v_B + s_B v_A \right)$$

It is given that $v_A = 0.4$ m/s. Substituting the value into the velocity constraint equation, we have

$$v_B = -0.4 \text{ m/s}$$

agreeing with our intuition.

Differentiating the general velocity relationship to obtain the acceleration relation which shall be incorporated with the result from the kinetic analysis:

$$0 = v_A^2 + s_A a_A + v_B^2 + s_B a_B - \cos 150 \left(s_A a_B + s_B a_A + 2 v_A v_B \right)$$

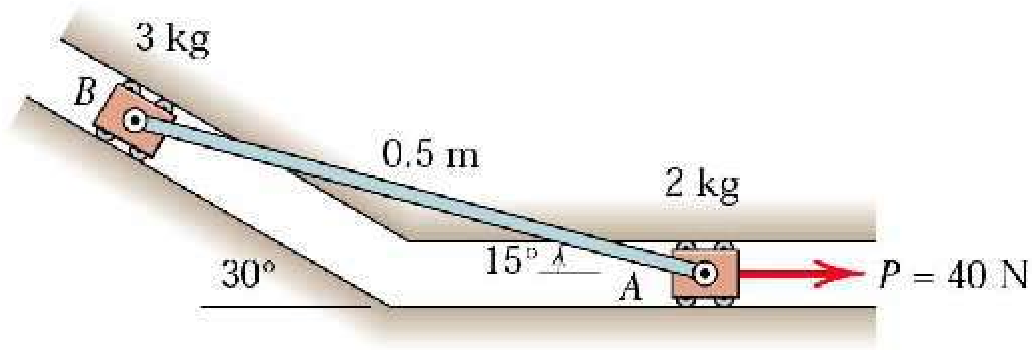


Figure 8.15: Example 8.8 ([3], pp. 138)

Substitute all known kinematical parameters into the equation, we have the relationship between a_A and a_B at this instant as

$$0 = 0.04287 + 0.4829a_A + 0.4829a_B$$

Kinetic analysis starts with the free body diagrams of both carts, which are simple because of the assumption of light rigid bar and negligible friction. Also, the system is oriented to lie in a horizontal plane which makes the gravity force pointing perpendicularly to the paper. Hence it will not affect the planar motion of the carts. Applying Newton's law onto each cart along its traveling direction, we have

$$[\Sigma F = ma] \qquad 40 - T\cos 15 = 2a_A$$

$$-T\cos 15 = 3a_B$$

Combining both equations of motion with the constraint equation developed above, we may solve for the tension and the accelerations

$$a_A = 7.95 \ \ \text{m/s}^2, \quad a_B = -8.04 \ \ \text{m/s}^2, \quad T = 25.0 \text{ N}$$

Example 8.9 ([4], Prob. 3/46) With the blocks initially at rest, the force P is increased slowly from zero to 260 N. Plot the accelerations of both masses as functions of P.

Solution: Free body diagram of both blocks are sketched in fig. 8.18. First, we calculate the normal forces, the static, and the kinetic frictions;

$$N_A = 35g, \quad N_B = N_A + 42g = 77g$$

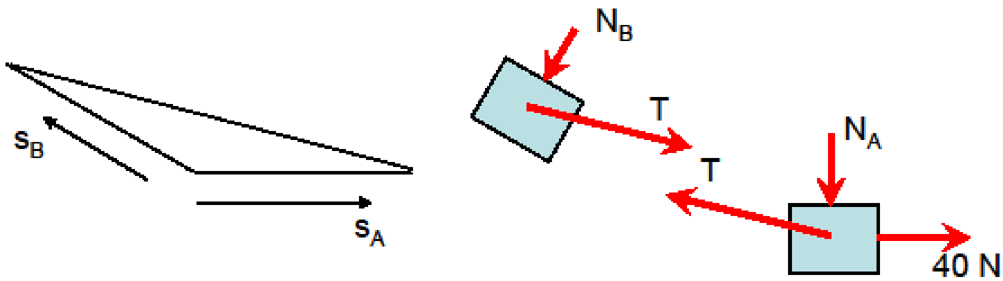


Figure 8.16: Solution to example 8.8

$$F_{A_{\max}} = 0.2N_A = 68.67 \text{ N}, \quad F_{B_{\max}} = 0.15N_B = 113.3 \text{ N}$$
$$F_{A_k} = 0.15N_A = 51.5 \text{ N}, \quad F_{B_k} = 0.10N_B = 75.54 \text{ N}$$

Next, we need to think of all possible situations of the motion of A and B. There are totally 5 situations. Three of them are (a) no motion at all, (b) B and A move together, and (c) B and A move separately. However, there are two situations which are not possible. They are (d) B moves alone. This is because F_A will be developed to resist the motion. Hence it makes A to move eventually. Another impossible situation is that (e) A moves alone. This is because the force P is applied at block B in a slowly increasing fashion from zero. So it is unlikely that F_A will suddenly jump right to $F_{A_{\max}}$, making block A to jerk and move immediately while leaving block B immobile due to the counter-action of the friction on the applied force.

To study the motion of both masses as functions of slowly increasing P, we reason as follow;

1. For the applied force $0 \leq P \leq F_{B_{\max}}$, i.e. the applied force is less than the maximum supplied friction force, the friction F_B will be developed accordingly to cancel with the applied force P. Hence F_A will stay zero and so both blocks will not move. Therefore for $0 \leq P \leq 113.3$ N, $a_A = 0$ and $a_B = 0$.

2. If we increase the applied force beyond $F_{B_{\max}}$ but below some value, block A and B will go together as a single body. For this to be the case, it is necessary that $F_A \leq F_{A_{\max}}$ (so sliding between A and B is not yet happen) and $F_B = F_{B_k}$ (so block B moves). Substitute these friction values into the equation of motions, we have

$$[\Sigma F = ma] \qquad F_A = 35a$$

$$P - F_A - F_{B_k} = 42a$$

We may use these equations in determining the range of P that makes two blocks go together. When P increases to the value of $F_{B_{\max}} = 113.3$ N, F_B will drop from $F_{B_{\max}} = 113.3$ N to $F_{B_k} = 75.54$ N. Block B will then start to move, for which we may apply the above equations of motion:

$$F_A = 35a$$

$$113.3 - F_A - 75.54 = 42a$$

Consequently,

$$a = 0.49 \ \ \mathrm{m/s^2}, \quad F_A = 17.16 \ \mathrm{N}$$

Note the values jump from zero to kick the blocks off from the resting state.

To determine the upper bound of P that block A and B still move together, we use the fact that F_A will be developed to $F_{A_{\max}}$ (so they are about to slip relative to each other) when P reaches the upper limit. We may recall the above equations of motion again:

$$68.67 = 35a$$

$$P - 68.67 - 75.54 = 42a$$

Consequently,

$$P = 226.6 \ \mathrm{N}, \quad a = 1.962 \ \ \mathrm{m/s^2}$$

Between these extremum values, we may solve for the (common) acceleration directly by viewing both blocks as a new single block of 77 kg. Hence, the internal friction force F_A vanishes and the acceleration

$[\Sigma F = ma]$ $$P - F_{B_k} = 77a, \quad a = \tfrac{P-75.54}{77}$$

which is a linear function of P in the range of $113.3 \leq P \leq 226.6$ N.

3. If we further increase the applied force P beyond 226.6 N, block A will slide backward relative to B because increasing P will make B accelerates more and more. Since A slips, F_A will now drop to F_{A_k}. Re-applying the equations of motion again,

$$51.5 = 35a_A$$

$$P - 51.5 - 75.54 = 42a_B$$

Therefore, with the applied force $226.6 < P \leq 260$ N, block A will move with constant acceleration $a_A = 1.47 \ \ \mathrm{m/s^2}$ by the kinetic friction. The applied force P will affect only the acceleration of block B. The exact expression of it is

$$a_B = \frac{P - 127.04}{42}$$

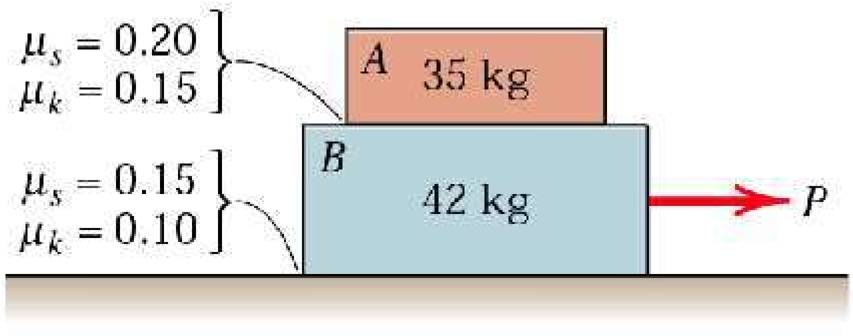


Figure 8.17: Example 8.9 ([4], pp. 139)

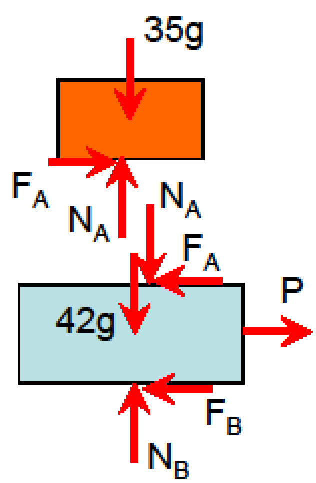


Figure 8.18: Solution to example 8.9

which is a linear function of P. Plotting the functions of the accelerations with respect to applied force, depicted in fig. 8.19, we see the jump in acceleration again caused by the friction transition dropping from impending to kinetic value.

Example 8.10 ([4], Prob. 3/47) The system is released from rest in the position shown. Calculate the tension T in the cord and the acceleration a of the 30-kg block. The small pulley attached to the block has negligible mass and friction. (Suggestion: First establish the kinematic relationship between the accelerations of the two bodies.)

Solution: We see that motion of both blocks are related by the cable-pulley mechanism. Definitions of kinematic parameters are illustrated in fig. 8.21. The

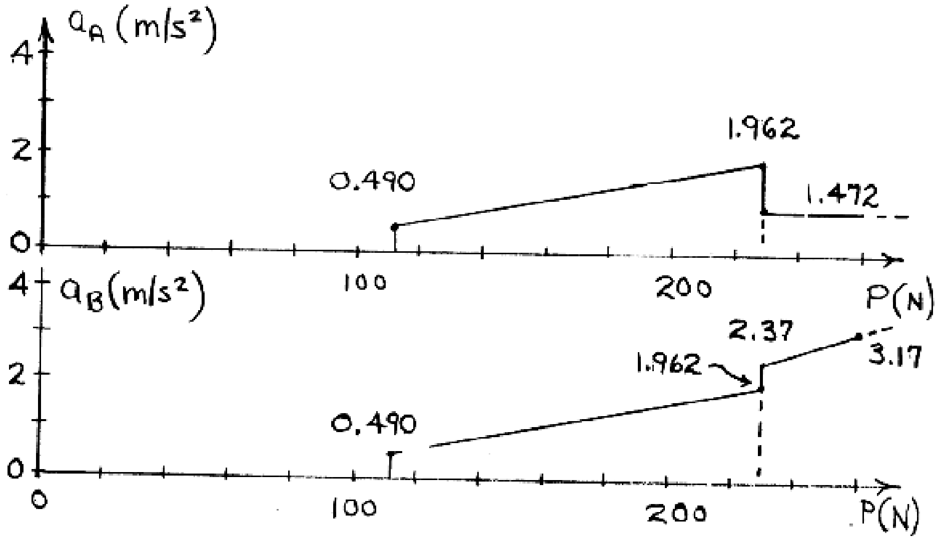


Figure 8.19: The acceleration of both masses

sliding block displacement x causes the change in the hypotenuse b of the right triangle. In turn, this will make the hanging block to move, denoted as the displacement y, according to the unchanged cable length. Mathematically,

$$b^2 = c^2 + x^2$$

$$b + y = l$$

where c and l are some positive constants. Differentiating the relationship to obtain the acceleration equations;

$$b\dot{b} = x\dot{x} \quad \text{and} \quad \dot{b} + \dot{y} = 0$$

$$\dot{b}^2 + b\ddot{b} = \dot{x}^2 + x\ddot{x} \quad \text{and} \quad \ddot{b} + \ddot{y} = 0$$

At this instant, $\frac{x}{b} = \frac{4}{5}$ and $\dot{x} = 0$, $\dot{b} = 0$ since the system is released from rest. Substituting these values into the constraint, we have

$$\frac{\ddot{b}}{\ddot{x}} = \frac{4}{5}$$

Next, the kinetics aspect will be analyzed. Since the system involves friction, of which its direction depends on the sliding direction, we assume that the cylinder block moves downward. Hence the sliding block will move to the left due to the constraint which causes the friction force to point rightward. Based on this assumption, free body diagram of the objects may be drawn, as depicted in fig. 8.21.

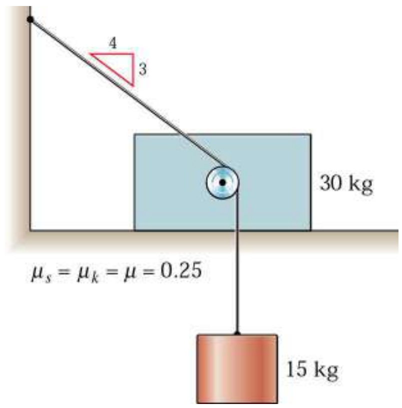


Figure 8.20: Example 8.10 ([4], pp. 139)

Stating Newton's law on the 30-kg block, we have

$[\Sigma F_y = ma_y] \qquad T \times \frac{3}{5} - T - 30g + N = 0, \quad N = 30g + \frac{2}{5}T$

$[\Sigma F_x = ma_x] \qquad F - T \times \frac{4}{5} = 0.25N - T \times \frac{4}{5} = 30\ddot{x}$

For the 15-kg cylinder, its equation of motion is

$[\Sigma F_y = ma_y] \qquad 15g - T = 15\ddot{y} = -15\ddot{b}$

Now we have two effective equations for three unknowns: T, $\ddot{x}$, and $\ddot{b}$. One more equation comes from the kinematic constraint. Substituting the acceleration of b and x obtained from the equations of motion into it, we may solve for the tension;

$$\frac{4}{5} = \frac{\ddot{b}}{\ddot{x}} = \frac{(T/15 - g) \times 30}{7.5g - 0.7T}, \quad T = 137.9 \text{ N}$$

And the acceleration of the block becomes

$$\ddot{x} = \frac{7.5g - 0.7T}{30} = -0.766 \ \ \text{m/s}^2$$

8.5 Curvilinear Motion

There are choices for the coordinate systems used in describing the curvilinear motion, as studied in chapter 7. One may first need to decide the appropriate one, depending mainly on the information given and seeking. Then, the kinematical

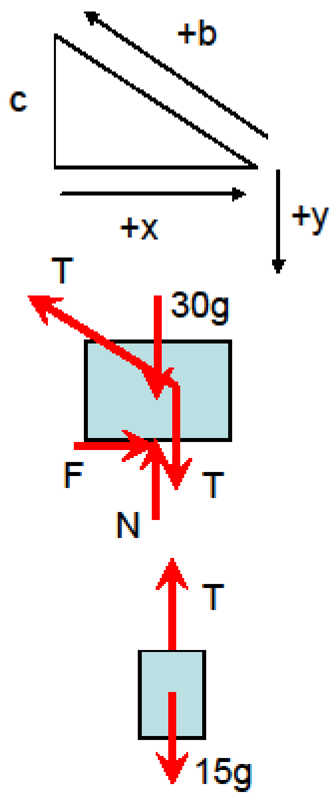


Figure 8.21: Solution to example 8.10

analysis must be carried out even its related quantities have not been asked for. Note that the complete results might not yet be obtained.

After that, one may draw the helpful free body diagram and set up Newton's law along the axes of the selected coordinate frame. For typical ones, they are

x-y coordinate system:

$$\begin{aligned}\Sigma F_x &= m\ddot{x} \\ \Sigma F_y &= m\ddot{y}\end{aligned} \tag{8.2}$$

n-t coordinate system:

$$\begin{aligned}\Sigma F_n &= m\left(\rho\dot{\beta}^2\right) = m\left(v^2/\rho\right) \\ \Sigma F_t &= m\dot{v}\end{aligned} \tag{8.3}$$

r-θ coordinate system:

$$\begin{aligned}\Sigma F_r &= m\left(\ddot{r} - r\dot{\theta}^2\right) \\ \Sigma F_\theta &= m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\end{aligned} \tag{8.4}$$

As the last word, the positive sense of the force and accceleration must be consistent.

Example 8.11 ([3], Prob. 3/58) The member OArotates about a horizontal axis through O with a constant counterclockwise velocity $\omega = 3$ rad/s. As it passes the position $\theta = 0$, a small block of mass m is placed on it at a radial distance $r = 450$ mm. If the block is observed to slip at $\theta = 50°$, determine the coefficent of static friction μ_s between the block and the member.

Solution: While either r-θ or n-t coordinate system is equally suit for this problem, the latter shall be adopted. Hence, the kinematic parameters right at the time of slipping in n-t frame notation may be expressed as

$$\rho = 0.45 \text{ m}, \quad \dot{\rho} = 0$$

$$\beta = 50°, \quad \dot{\beta} = 3 \text{ rad/s}, \quad \ddot{\beta} = 0$$

Note that $\dot{\rho} = 0$ because the block is not yet slipping. However $\ddot{\rho} \neq 0$.

Free body diagram of the block is shown in fig. 8.23. Here it is assumed that the block slides down relative to the member and hence the friction points upward. Setting up the equations of motion and substituting the known kinematic parameters, we have

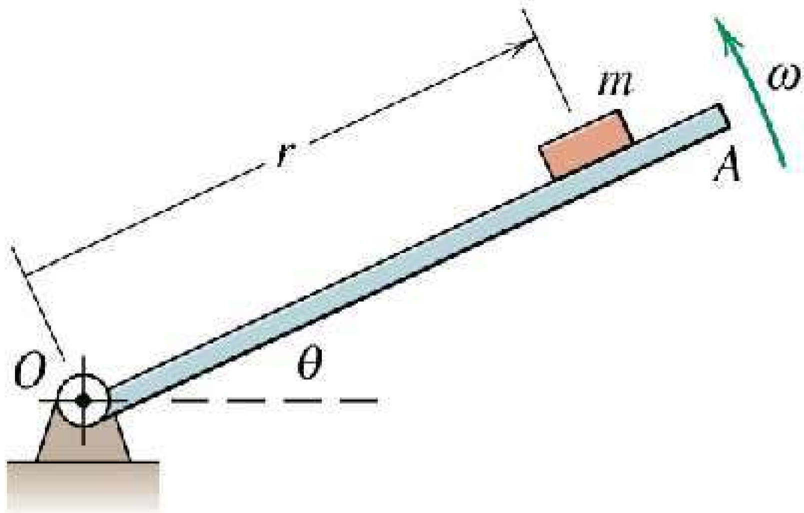


Figure 8.22: Example 8.11 ([3], pp. 145)

$[\Sigma F_t = ma_t]$ $$N - mg\cos 50 = m(\dot{\rho}\dot{\beta} + \rho\ddot{\beta}) = 0, \quad N = mg\cos 50$$

$[\Sigma F_n = ma_n]$ $$mg\sin 50 - F = m(\rho\dot{\beta}^2)$$

At the slipping moment, $F = F_s = \mu_s N$. From the second equation, direction of the friction must be upward since the gravity force ($mg\sin 50$) is greater than the inertial force ($m\rho\dot{\beta}^2$). Hence the assumed situation is correct. This means the bar OA rotates too slow than required to keep the block stay at rest on the bar. Friction will be developed to resist the block from sliding downward (i.e. to match the resultant force with the inertial force).

Substitute the static friction value in the above equation, we may solve for the friction coefficient.

$$mg\sin 50 - \mu_s mg\cos 50 = m\rho\dot{\beta}^2, \quad \mu_s = 0.549$$

Example 8.12 ([3], Prob. 3/69) A 2-kg sphere S is being moved in a vertical plane by a robotic arm. When the arm angle θ is 30°, its angular velocity about a horizontal axis through O is 50 deg/s CW and its angular acceleration is 200 deg/s^2 CCW. In addition, the hydraulic element is being shortened at the constant rate of 500 mm/s. Determine the necessary minimum gripping force P if the coefficient of static friction between the sphere and the gripping surfaces is 0.5. Compare P to the minimum gripping force P_s required to hold the sphere in static equilibrium in the 30° position.

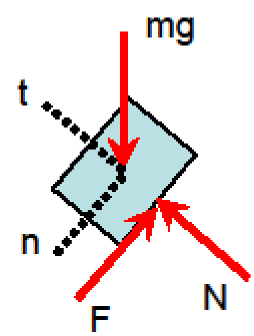


Figure 8.23: Solution to example 8.11

Solution: We will use r-θ coordinates in describing the motion of the sphere. The frame is attached to the ball in the free body diagram as shown in fig. 8.25. The unknowns are components of the friction force, F_r and F_θ. From the specified kinematics, we may write

$$r = 1 \text{ m}, \quad \dot{r} = -0.5 \text{ m/s}, \quad \ddot{r} = 0$$

$$\theta = 30^\circ, \quad \dot{\theta} = -50 \times \frac{\pi}{180} = -0.873 \text{ rad/s}, \quad \ddot{\theta} = 200 \times \frac{\pi}{180} = 3.49 \text{ rad/s}^2$$

Setting up the equations of motion in r-θ frame,

$$[\Sigma F_r = ma_r] \qquad -mg\sin\theta + 2F_r = m(\ddot{r} - r\dot{\theta}^2), \quad F_r = 4.143 \text{ N}$$

$$[\Sigma F_\theta = ma_\theta] \qquad 2F_\theta - mg\cos\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}), \quad F_\theta = 12.859 \text{ N}$$

The components are combined to obtain the friction force which, by the minimum gripping force condition, is the static friction. Therefore,

$$F = \sqrt{F_r^2 + F_\theta^2} = 13.51 = \mu_s P = 0.5P, \quad P = 27.02 \text{ N}$$

If the robot is not moving at the same posture, the required minimum gripping force will be generated purely just to cancel the gravity force, as depicted in fig. 8.25. The friction force in that case may be determined as

$$2F_s = 2\mu_s P_s = 2 \times 0.5P_s = mg, \quad P_s = 19.62 \text{ N}$$

which is less than the moving arm case, as naturally expected.

Example 8.13 ([4], Prob. 3/78) A flatbed truck going 100 km/h rounds a horizontal curve of 300 m radius inwardly banked at 10°. The coefficient of static friction between the truck bed and the 200 kg crate it carries is 0.70. Calculate

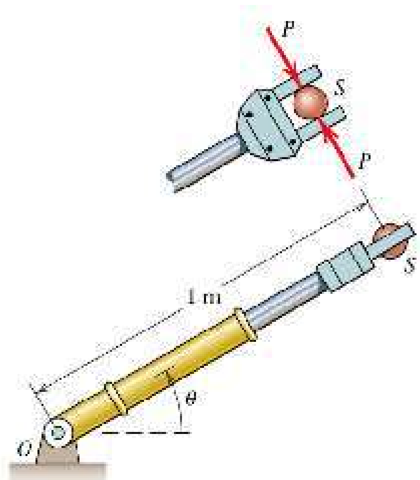


Figure 8.24: Example 8.12 ([3], pp. 148)

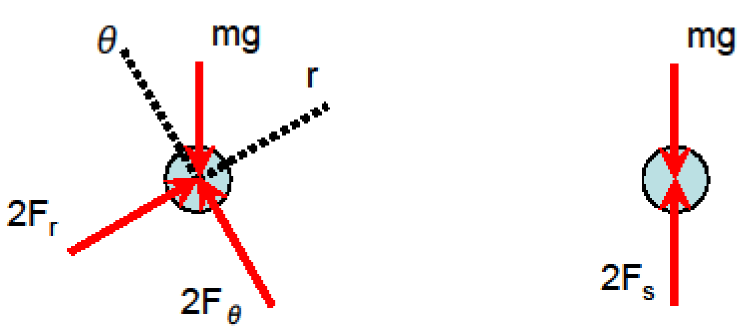


Figure 8.25: Solution to example 8.12

the friction force F acting on the crate.

Solution: The crate travels with the truck in a horizontal-planar circular motion and so its admissible acceleration are in n-t directions, pointing toward the center of the curve and perperdicular to the drawing of its free body diagram, as illustrated in fig. 8.27. To meet these inertial forces, friction force has to be directed down the slope. This implies the crate *tends* to slide up the truck bed (being fling away).

In setting up the equation of motion, first we cast Newton's second law along the normal direction because the associated acceleration is known.

$$[\Sigma F_n = mv^2/\rho] \qquad N \sin 10 + F \cos 10 = \tfrac{m}{300}\left(100 \times \tfrac{10}{36}\right)^2$$

Other direction that we know its motion is the vertical direction. Since it is not moving,

$$[\Sigma F_y = 0] \qquad -mg + N \cos 10 - F \sin 10 = 0$$

Consequently, the developed normal and friction force would be

$$N = 2021.52 \text{ N}, \quad F = 165.9 \text{ N}$$

It is necessary to check if this friction can be actually generated:

$$F_{\max} = 0.7N = 1415 \text{ N} > F$$

Hence the required friction force of 165.9 N will be provided. Therefore the crate *tends* to slide up due to high speed circular motion. However, it is still to far from sliding up. We can increase the truck speed yet the crate does not move relative to the truck bed.

Example 8.14 ([3], Prob. 3/79) The flatbed truck starts from rest on a road whose constant radius of curvature is 30 m and whose bank angle is 10°. If the constant forward acceleration of the truck is 2 m/s^2, determine the time t after the start of motion at which the crate on the bed begins to slide. The coefficient of static friction between the crate and the truck bed is $\mu_s = 0.3$, and the truck motion occurs in a horizontal plane.

Solution: First, free body diagrams of the crate for the static and dynamic case are drawn in fig. 8.29. After applying the equilibrium condition for the static case, we may solve for the normal and friction force:

$$[\Sigma F_{y'} = 0] \qquad N_s = 200g \cos 10 = 1932.2 \text{ N}$$

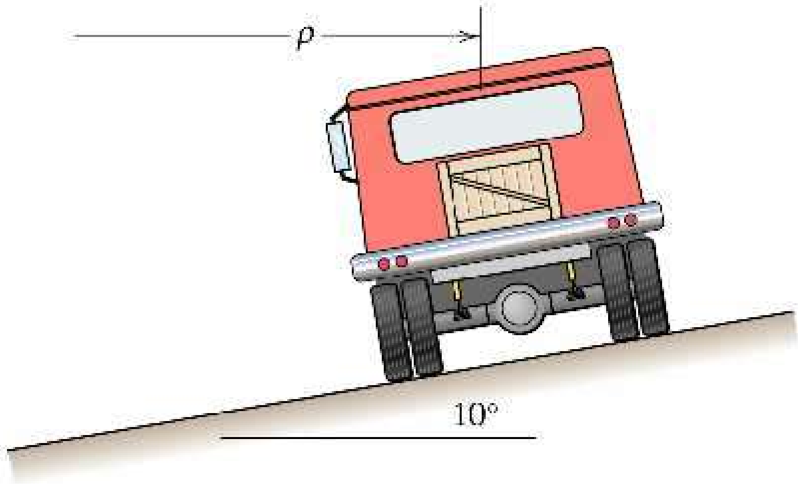


Figure 8.26: Example 8.13 ([4], pp. 150)

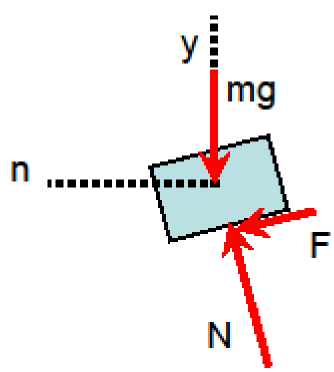


Figure 8.27: Solution to example 8.13

$[\Sigma F_{x'} = 0]$ $$F = 200g \sin 10 = 340.7 \text{ N}$$

upward to prevent sliding down the incline. We need to check whether this friction force is actually realizable. The maximum static friction force is

$$F_s = 0.3N_s = 579.66 \text{ N}$$

which is greater than the required friction.

When the crate is moving, the friction will not lie solely along the x'-direction because the crate is moving along the horizontal-circular path. Nevertheless, when it starts to slide, the magnitude of the friction force must be equal to $F_s = 0.3N$. Therefore, we decompose the friction force along the x'- (downward the truckbed to match the component of the normal inertial force down the incline) and t-axis (pointing inward the paper) in association with the natural axes of the crate motion relative to the truckbed. The components are named F_{sn} and F_{st} respectively. See fig. 8.29.

Specified kinematic parameters of the crate may be expressed in n-t coordinates as

$$\rho = 30 \text{ m}, \quad \dot{\rho} = 0, \quad \ddot{\rho} = 0$$
$$a_t = 2 \ \text{ m/s}^2$$

from which the velocity may be deduced:

$[a_t = \dot{v}]$ $$v = a_t t = 2t$$

Next, applying Newton's law along the y-, n-, and t-direction to arrive at the equations of motion.

$[\Sigma F_y = 0]$ $$N \cos 10 - 200g - F_{sn} \sin 10 = 0$$

$[\Sigma F_n = mv^2/\rho]$ $$F_{sn} \cos 10 + N \sin 10 = 200 \times \left(\frac{4t^2}{30}\right)$$

$[\Sigma F_t = ma_t]$ $$F_{st} = 200 \times 2 = 400 \text{ N}$$

There are three unknowns, namely N, F_{sn}, and t, with two effective equations. The third equation may be formulated by recognizing that the magnitude of the friction must be equal to the static value when the crate begins sliding.

$[F_{sn}^2 + F_{st}^2 = F_s^2]$ $$F_{sn}^2 + 400^2 = (0.3N)^2$$

$$F_{sn} = \sqrt{0.09N^2 - 160000}$$

Substitute the expression into the equilibrium equation, the normal force

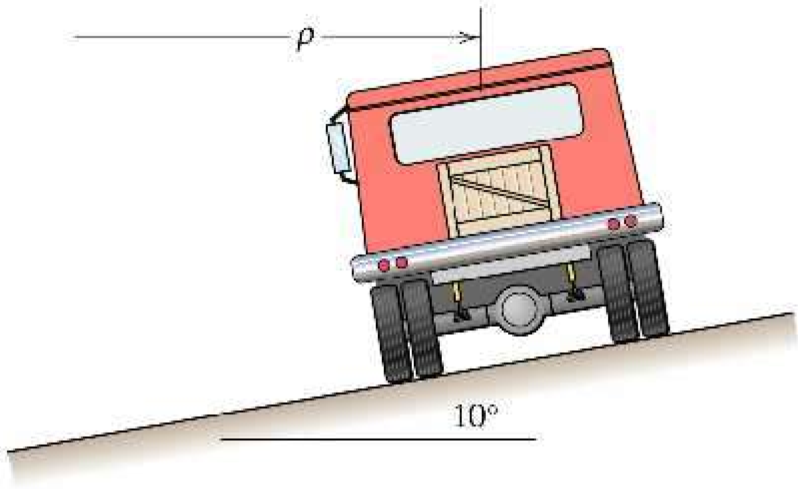


Figure 8.28: Example 8.14 ([3], pp. 150)

may now be determined.

$$N = 2076.47, 1919.24 \text{ N}$$

When the truck enters the curve, the normal acceleration is developed. From the free body diagram, it is necessary that $N > N_s$ to match the positive component of the normal inertial force perpendicular to the incline. Therefore

$$N = 2076.47 \text{ N} > N_s$$

The other variables may now be determined.

$$F_{sn} = \sqrt{0.09 \times 2076.47^2 - 160000} = 477.55 \text{ N}$$

$$F_{st} = 400 \text{ N}$$

$$477.55 \cos 10 + 2076.47 \sin 10 = 200 \times \left(\frac{4t^2}{30}\right), \quad t = 5.58 \text{ sec}$$

Example 8.15 ([3], Prob. 3/77) The small object is placed on the inner surface of the conical dish at the radius shown. If the coefficient of static friction between the object and the conical surface is 0.30, for what range of angular velocities ω about the vertical axis will the block remain on the dish without slipping? Assume that speed changes are made slowly so that any angular acceleration may be neglected.

Solution: Figure 8.31 shows free body diagrams of the problem for two

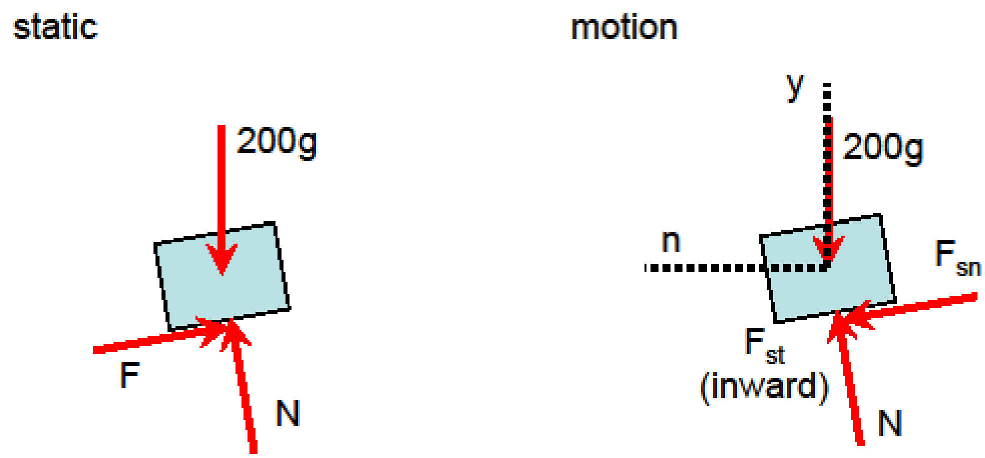


Figure 8.29: Solution to example 8.14

cases, i.e. on the verge of slipping downward and upward for minimum and maximum allowable angular velocity. Because the angular acceleration may be neglected, the total static friction force ($0.3N$) points along the conical dish as denoted by F_s. Their directions are oriented in a way to match with the magnitude of the centerfugal acceleration.

With the specified kinematic parameters, we have

$$\rho = 0.2 \text{ m}, \quad \dot{\rho} = 0, \quad \ddot{\rho} = 0$$

For the minimum angular velocity, Newton's law allows us to write

$$[\Sigma F_y = 0] \qquad N \cos 30 + 0.3N \sin 30 - mg = 0$$

$$[\Sigma F_n = mr\omega^2] \qquad N \sin 30 - 0.3N \cos 30 = m(0.2\omega_{\text{min}}^2)$$

for which the minimum angular velocity becomes

$$\omega_{\text{min}} = 3.405 \quad \text{rad/s}$$

If the maximum angular velocity is applied, the equations of motion slightly change to

$$[\Sigma F_y = 0] \qquad N \cos 30 - 0.3N \sin 30 - mg = 0$$

$$[\Sigma F_n = mr\omega^2] \qquad N \sin 30 + 0.3N \cos 30 = m(0.2\omega_{\text{max}}^2)$$

for which the maximum angular velocity is

$$\omega_{\text{max}} = 7.214 \quad \text{rad/s}$$

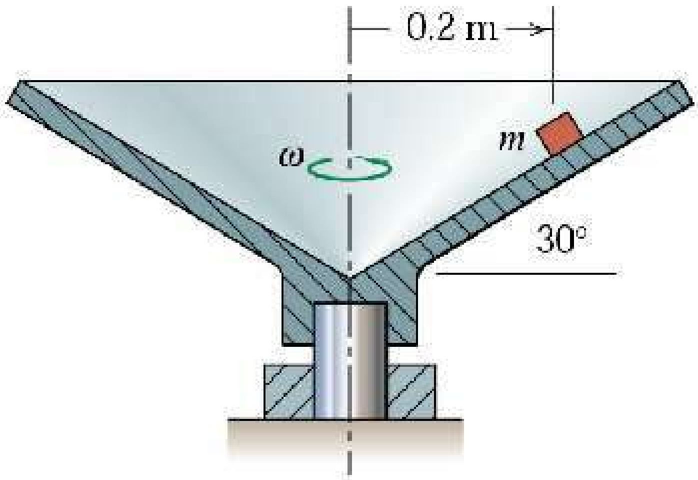


Figure 8.30: Example 8.15 ([3], pp. 150)

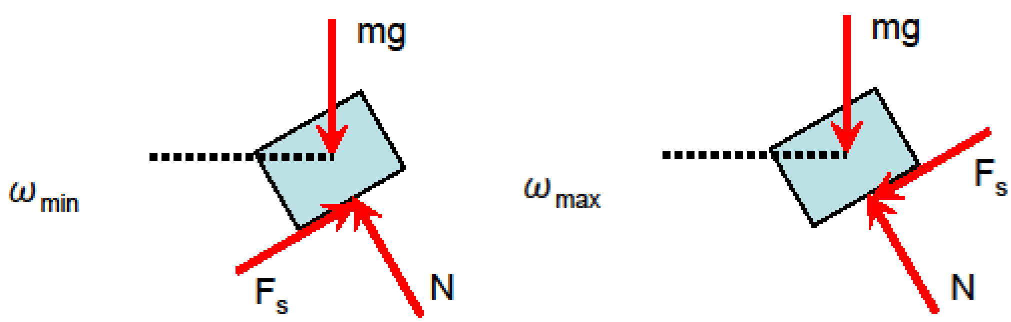


Figure 8.31: Solution to example 8.15

Therefore the angular velocity value between $3.405 < \omega < 7.214$ rad/s will not make the block sliding.

Example 8.16 ([4], Prob. 3/82) The 2-kg slider fits loosely in the smooth slot of the disk, which rotates about a vertical axis through point O. The slider is free to move slightly along the slot before one of the wires becomes taut. If the disk starts from rest at time $t = 0$ and has a constant clockwise angular acceleration of 0.5 rad/s^2, plot the tensions in wires 1 and 2 and the magnitude N of the force normal to the slot as functions of time t for the interval $0 \leq t \leq 5$ sec.

Solution: Kinematic will be performed prior to kinetic analysis. Since the disc is rotating with constant clockwise angular acceleration of $\ddot{\theta} = 0.5$ rad/s^2, with the zero initial conditions, the velocity and angle-time function become

$$\dot{\theta} = 0.5t, \quad \theta = 0.25t^2$$

From the installation, $r = 0.1$ m. Because the slider is constrained by the strings to move *slightly* with respect to the disc, its motion may be treated as same as the disk. Hence

$$\dot{r} = 0, \quad \ddot{r} = 0$$

Either one of the strings may be taut at an instant. So we assume that the normal and tension force be acting along the indicated direction as depicted in the free body diagram (fig. 8.33). This implies the first string being tensioned and the second one slacked. Using the r-θ coordinate frame where O is the reference point and the vertical line the reference direction, we may set up the equations of motion as follow.

$$\left[\Sigma F_r = m(\ddot{r} - r\dot{\theta}^2)\right] \quad -N\cos 45 - T\cos 45 = 2\left(-0.1 \times (0.5t)^2\right) = -0.05t^2$$

$$\left[\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\right] \quad N\sin 45 - T\sin 45 = 2\,(0.1 \times 0.5) = 0.1$$

Thanks to the use of moving coordinate frame, we don't have to worry about the change in the direction of the normal and tension forces. We now can solve for their values as a function of time.

$$N = \frac{0.05t^2 + 0.1}{\sqrt{2}}, \quad T = \frac{0.05t^2 - 0.1}{\sqrt{2}}$$

From the expressions, the normal force is always positive. Hence the assumed direction is correct. For the tension force, it will be negative for $t < 1.414$ sec. This is in accordance with our intuition that the inertial effect of the slider at the beginning makes the second string to pull it along with the disc motion. After that the slider motion will lead the disc, causing the first string being pulled. In turn the second string becomes slack. In summary,

$$T_1 = \begin{cases} 0, & 0 \le t \le 1.414 \text{ s} \\ \frac{0.05t^2 - 0.1}{\sqrt{2}}, & t > 1.414 \text{ s} \end{cases}$$

$$T_2 = \begin{cases} \frac{0.1 - 0.05t^2}{\sqrt{2}}, & 0 \le t \le 1.414 \text{ s} \\ 0, & t > 1.414 \text{ s} \end{cases}$$

Plots of the tensions and normal force are shown in fig. 8.34.

Example 8.17 ([4], Prob. 3/89) A small rocket-propelled vehicle of mass m travels down the circular path of effective radius r under the action of its weight and a constant thrust T from its rocket motor. If the vehicle starts from rest at

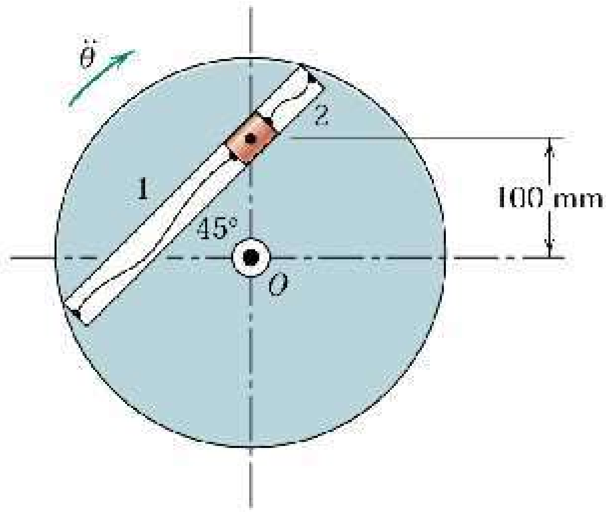


Figure 8.32: Example 8.16 ([4], pp. 151)

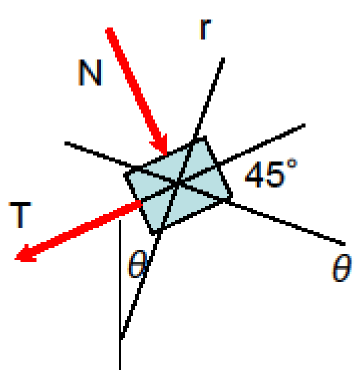


Figure 8.33: Solution to example 8.16

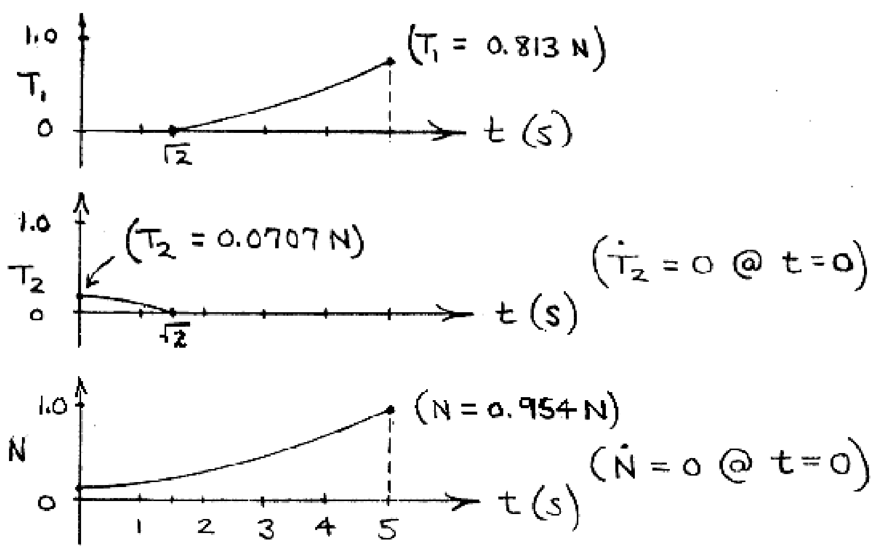


Figure 8.34: Plots of tension wires and normal force acting on the slider

A, determine its speed v when it reaches B and the magnitude N of the force exerted by the guide on the wheels just prior to reaching B. Neglect any friction and any loss of mass of the rocket.

Solution: Free body diagram of the vehicle showing the gravity force, the normal force, and the propulsion force in action is depicted in fig. 8.36. Because the vehicle is moving along the constrained circular path, we decide to use n-t coordinate frame. Newton's law then gives the following governing equations:

$$[\Sigma F_n = ma_n] \qquad N - mg\sin\theta = mv^2/r$$

$$[\Sigma F_t = ma_t] \qquad T + mg\cos\theta = ma_t, \quad a_t = \tfrac{T+mg\cos\theta}{m}$$

The unknowns are the normal force and velocity, which may be determined from the velocity-acceleration relationship. With the vehicle starts from rest at A, we have

$$[vdv = a_t ds] \qquad v^2/2 = \textstyle\int_0^\theta a_t(rd\theta)$$

Substituting the value of tangential acceleration, the velocity written in terms of the traveling angle θ is

$$v^2 = 2r\left(\frac{T\theta}{m} + g\sin\theta\right)$$

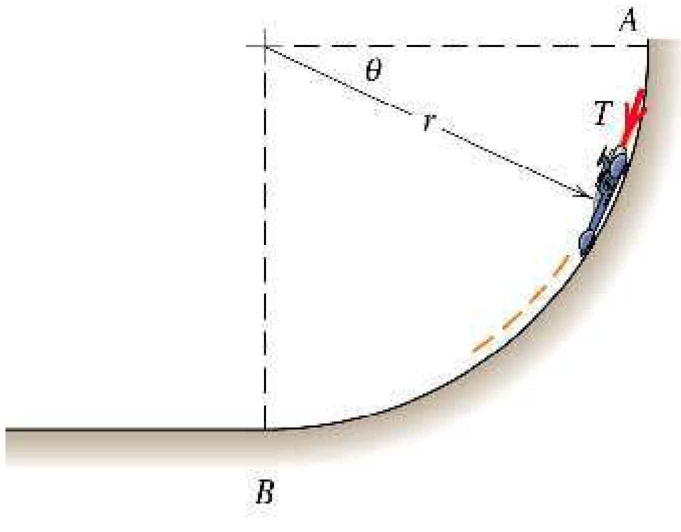


Figure 8.35: Example 8.17 ([4], pp. 153)

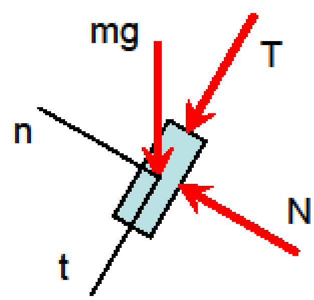


Figure 8.36: Solution to example 8.17

Conseqently, the normal force is

$$N = 3mg\sin\theta + 2T\theta$$

At B, $\theta = \frac{\pi}{2}$. The corresponding velocity and normal force are

$$v_{\theta=\frac{\pi}{2}} = \sqrt{r\left(\frac{\pi T}{m} + 2g\right)}$$

$$N_{\theta=\frac{\pi}{2}} = 3mg + T\pi$$

Example 8.18 ([3], Prob. 3/100) A hollow tube rotates about the horizontal axis through point O with constant angular velocity ω_o. A particle of mass m is introduced with zero relative velocity at $r = 0$ when $\theta = 0$ and slides outward through the smooth tube. Determine r as a function of θ.

Solution: The angular motion of the tube may be expressed as

$$\dot{\theta} = \omega_o, \quad \ddot{\theta} = 0$$

Initially, $r = 0$, $\dot{r} = 0$, and $\theta = 0$. Therefore the tube angle as a function of time is simply

$$\theta = \omega_o t$$

Figure 8.38 shows the free body diagram of the mass at an angle θ. To determine $r(t)$, we formulate the equation of motion along r-direction.

$$[\Sigma F_r = ma_r] \qquad mg\sin\theta = m(\ddot{r} - r\dot{\theta}^2)$$

Substitute the angle function into it, the differential equation of $r(t)$ is obtained:

$$\ddot{r} - \omega_o^2 r = g\sin\omega_o t$$

General solution $r(t)$ is the sum of the particular and homogeneous solution.

$$r(t) = r_p + r_h$$

Since the forcing function is sinusoidal function, it is known that the forced response of such linear system will also be sinusoidal function of the same frequency. It is obvious that a family of the particular solution has the form

$$r_p(t) = Cg\sin\omega_o t$$

Substitute the solution into the differential equation, parameter C may be determined.

$$-C\omega_o^2 g\sin\omega_o t - C\omega_o^2 g\sin\omega_o t = g\sin\omega_o t, \quad C = -\frac{1}{2\omega_o^2}$$

Homogeneous solution is the solution of the equation when the forcing function is null. For this problem, we look for the solution, $r_h(t)$, of

$$\ddot{r} - \omega_o^2 r = 0$$

which has the well known form of the exponential function:

$$r_h(t) = Ae^{st}$$

for specific values of s and A.

The value of s, known as the characteristic root, may be determined by substituting $r_h(t)$ back into the homogeneous equation.

$$As^2e^{st} - A\omega_o^2 e^{st} = 0, \quad s = \pm\omega_o$$

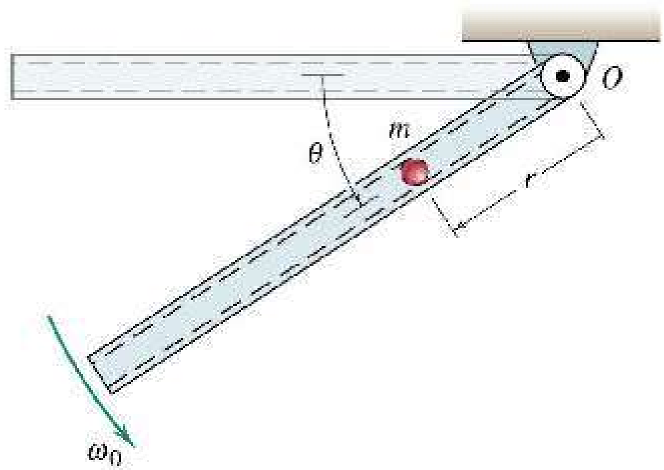


Figure 8.37: Example 8.18 ([3], pp. 156)

Therefore, the general homogeneous solution will be the linear combination of the exponential functions:

$$r_h(t) = Ae^{\omega_o t} + Be^{-\omega_o t}$$

Consequently, the general solution $r(t)$ will be

$$r(t) = r_p + r_h = Ae^{\omega_o t} + Be^{-\omega_o t} - \frac{1}{2\omega_o^2} g \sin \omega_o t$$

where the coefficient A and B are determined from the initial conditions:

$$r(0) = 0 = A + B$$

and

$$\dot{r}(0) = 0 = A\omega_o - B\omega_o - \frac{g}{2\omega_o}$$

Solving these simultaneous equations for A and B:

$$A = \frac{g}{4\omega_o^2}, \quad B = -\frac{g}{4\omega_o^2}$$

Substitute back into the general solution, we may determine the radial parameter as a function of time;

$$r(t) = \frac{g}{2\omega_o^2} (\sinh \theta - \sin \theta)$$

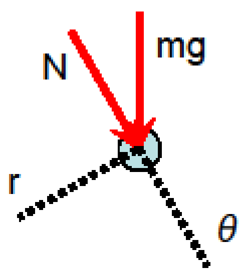


Figure 8.38: Solution to example 8.18

Example 8.19 ([4], Prob. 3/98) The small pendulum of mass m is suspended from a trolley that runs on a horizontal rail. The trolley and pendulum are initially at rest with $\theta = 0$. If the trolley is given a constant acceleration $a = g$, determine the maximum angle θ_{max} through which the pendulum swings. Also find the tension T in the cord in terms of θ.

Solution: Because the pendulum moves in circular motion *relative* to the trolley, we decide to use the moving n-t coordinate frame to describe its motion. With the constant radius of curvature, the acceleration components are simplified as shown in fig. 8.40.

To apply Newton's law, the absolute acceleration is needed. From the known acceleration of the trolley, that of the pendulum may be determined by the relative motion equation as

$$\left[\overline{a}_P = \overline{a}_C + \overline{a}_{P/C}\right] \qquad \overline{a}_P = g\mathbf{i} + l\dot{\theta}^2\mathbf{e}_n + l\ddot{\theta}\mathbf{e}_t$$

Equations of motion may now be formulated. Along the t-direction (where the unknown tension does not show up),

$$[\Sigma F_t = ma_t] \qquad -mg\sin\theta = m(-g\cos\theta + l\ddot{\theta})$$

Hence the angular acceleration of the pendulum, as a function of the angle, is

$$\ddot{\theta} = \frac{g}{l}(\cos\theta - \sin\theta)$$

which leads us to further solve for the angular velocity as

$$\left[\dot{\theta}d\dot{\theta} = \ddot{\theta}d\theta\right] \qquad \frac{\dot{\theta}^2}{2} = \int_0^\theta \frac{g}{l}(\cos\theta - \sin\theta)d\theta$$

$$\dot{\theta}^2 = \frac{2g}{l}(\sin\theta + \cos\theta - 1)$$

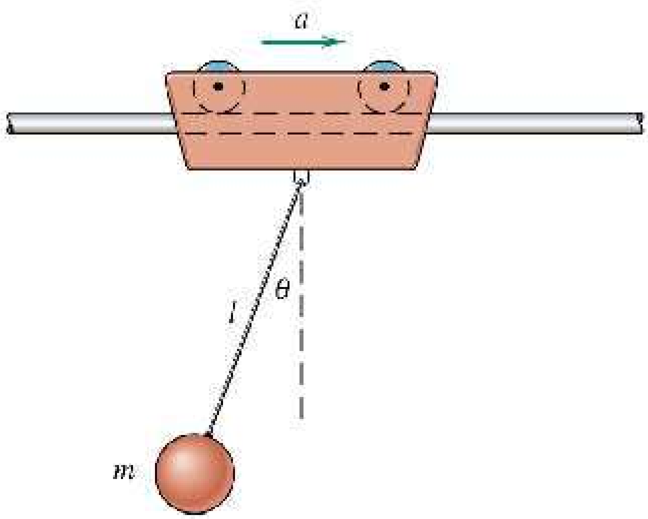


Figure 8.39: Example 8.19 ([4], pp. 155)

When the angle reaches the maximum or minimum value, its angular velocity will be zero. Applying this fact to the above equation, the necessary condition for being at the apex is

$$\sin\theta + \cos\theta = 1$$

For the maximum value, the equation yields $\theta_{\max} = \frac{\pi}{2}$.

Tension T in the cord may then be determined directly from solving the equation of motion in the normal direction:

$$[\Sigma F_n = ma_n] \qquad T - mg\cos\theta = m(g\sin\theta + l\dot{\theta}^2)$$

Substitute the velocity expression into the equation, the tension as a function of the swinging angle is

$$T = mg(3\sin\theta + 3\cos\theta - 2)$$

Example 8.20 ([4], Prob. 3/100) A small object is released from rest at A and slides with friction down the circular path. If the coefficient of friction is 0.2, determine the velocity of the object as it passes B. (Hint: Write the equations of motion in the n- and t- directions, eliminate N, and substitute $vdv = a_t r d\theta$. The resulting equation is a linear nonhomogeneous differential equation of the form $dy/dx + f(x)\,y = g(x)$, the solution of which is well known.)

Solution: Newton's law may be used to solve for the object's motion, known as

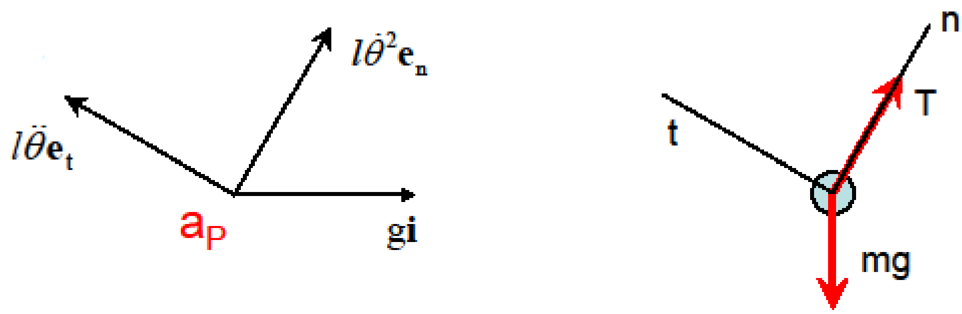


Figure 8.40: Solution to example 8.19

the inverse dynamic problem. Free body diagram ofthe sliding mass is drawn in fig. 8.42. Equations of motion along the normal and tangential direction are then

$$[\Sigma F_n = ma_n] \qquad N - mg\sin\theta = m(3\dot{\theta}^2)$$

$$[\Sigma F_t = ma_t] \qquad mg\cos\theta - 0.2N = m(3\ddot{\theta})$$

Since the normal force is uninterested, we eliminate it to obtain the differntial equation of the angle θ.

$$3\ddot{\theta} = g\cos\theta - 0.2(g\sin\theta + 3\dot{\theta}^2)$$

which is a *nonlinear* differential equation that is rather difficult to solve. By incorporating the kinematics relationship to the equation, we may transform the equation into the *linear* one and use the well-known technique in solving it. The velocity-acceleration relationship may be applied as follow.

$$\left[\dot{\theta}d\dot{\theta} = \ddot{\theta}d\theta\right] \qquad \dot{\theta}d\dot{\theta} = \tfrac{1}{2}d\left(\dot{\theta}^2\right) = \tfrac{1}{3}\left[g\cos\theta - 0.2(g\sin\theta + 3\dot{\theta}^2)\right]d\theta$$

Rearranging the equation to match the form $\frac{dy}{dx} + f(x)y = g(x)$, we have

$$\frac{d(\dot{\theta}^2)}{d\theta} + 0.4(\dot{\theta}^2) = \frac{2}{3}g\ (\cos\theta - 0.2\sin\theta)$$

which is a *linear* differential equation of $\dot{\theta}^2$ in terms of θ. To simplify the writing, let $\dot{\theta}^2 = u(\theta)$. We will now solve for $u(\theta)$.

General solution of $u(\theta)$ is the sum of the particular and homogeneous solution:

$$u(\theta) = u_p + u_h$$

$u_p(\theta)$ is the forced response of a sinusoidal function: $\frac{2}{3}g(\cos\theta - 0.2\sin\theta)$. Since the system is linear, the response will also be sinusoidal function of the same frequency. Therefore a family of the particular solution has the form

$$u_p(\theta) = A\cos\theta + B\sin\theta$$

Substitute the solution into the differential equation, parameters A and B may be determined.

$$-A\sin\theta + B\cos\theta + 0.4(A\cos\theta + B\sin\theta) = \frac{2}{3}g(\cos\theta - 0.2\sin\theta)$$

$$A = \frac{1.2}{3.48}g, \quad B = \left(\frac{2}{3} - \frac{0.48}{3.48}\right)g$$

Homogeneous solution is the solution of the equation when the forcing function is null. For this problem, we look for the solution, $u_h(\theta)$, of

$$\frac{du}{d\theta} + 0.4u = 0$$

which has the well known form of the exponential function:

$$u_h(\theta) = Ce^{s\theta}$$

for specific values of s and C.

The value of s, known as the characteristic root, may be determined by substituting $u_h(\theta)$ back into the homogeneous equation.

$$Cse^{s\theta} + 0.4Ce^{s\theta} = 0, \quad s = -0.4$$

Therefore, the general homogeneous solution becomes

$$u_h(\theta) = Ce^{-0.4\theta}$$

Consequently, the general solution $u(\theta)$ will be

$$u(\theta) = u_p + u_h = \frac{1.2}{3.48}g\cos\theta + \left(\frac{2}{3} - \frac{0.48}{3.48}\right)g\sin\theta + Ce^{-0.4\theta}$$

Because the object is released from rest, $u(0) = 0$. Applying this initial condition to the above equation, the constant C is

$$C = -\frac{1.2}{3.48}g$$

Hence, square of the time rate of change of θ is

$$u(\theta) = \dot{\theta}^2 = \frac{1.2}{3.48}g\cos\theta + \left(\frac{2}{3} - \frac{0.48}{3.48}\right)g\sin\theta - \frac{1.2}{3.48}ge^{-0.4\theta}$$

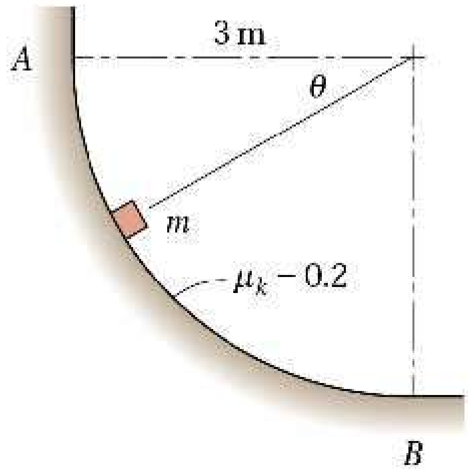


Figure 8.41: Example 8.20 ([4], pp. 156)

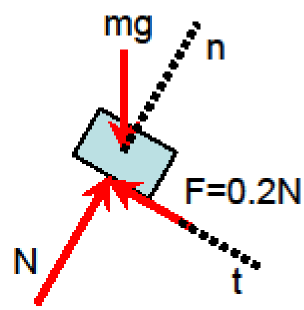


Figure 8.42: Solution to example 8.20

At $\theta = \frac{\pi}{2}$, $\dot{\theta}^2 = 3.382$. Therefore the block velocity following the circular path as it passes B is

$$\left[v = r\dot{\theta}\right] \qquad v_B = 3 \times \sqrt{3.382} = 5.52 \ \text{ m/s}$$

The account of friction force is the main source of complicated dynamics in this problem.

Example 8.21 ([3], Prob. 3/101) A small collar of mass m is given an initial velocity of magnitude v_o on the horizontal circular track fabricated from a slender rod. If the coefficient of kinetic friction is μ_k, determine the distance traveled before the collar comes to rest. (Hint: Recognize that the friction force depends on the net normal force.)

Solution: Free body diagram of the collar, depicted in fig. 8.44, reveals the forces acting on it. Note that the normal force is decomposed to N_v and N_h according to the natural coordinate frame n-t used. The friction force then lies along the t-axis. Setting up the equations of motion of the moving collar along y-, n-, and t-axes:

$$[\Sigma F_y = 0] \qquad N_v = mg$$

$$[\Sigma F_n = ma_n] \qquad N_h = mv^2/r$$

$$[\Sigma F_t = ma_t] \qquad -F = -\mu_k\sqrt{N_v^2 + N_h^2} = ma_t$$

To determine the distance traveled before the collar comes to rest, we think of the acceleration-velocity-displacement relationship. This formulation is convenient because it bypasses the internal time variable. With the known tangential acceleration after recognizing the normal force expressions, we may solve for the distance.

$$[vdv = a_t ds] \qquad vdv = -\tfrac{\mu_k}{mr}\sqrt{r^2m^2g^2 + m^2v^4}ds$$

Integrating the differential relation during the motion of the collar and assuming the distance traveled be s, we have

$$\int_{v_o}^{0}\frac{-r}{2\mu_k\sqrt{r^2g^2 + (v^2)^2}}d(v^2) = \int_0^s ds$$

$$s = \frac{r}{2\mu_k}\ln\left(\frac{v_o^2 + \sqrt{v_o^4 + r^2g^2}}{rg}\right)$$

Example 8.22 ([4], Prob. 3/101) The slotted arm OB rotates in a horizontal plane about point O of the fixed circular cam with constant angular velocity $\dot{\theta} = 15$ rad/s. The spring has a stiffness of 5 kN/m and is uncompressed when $\theta = 0$. The smooth roller A has a mass of 0.5 kg. Determine the normal force N that the cam exerts on A and also the force R exerted on A by the sides of the slot when $\theta = 45°$. All surfaces are smooth. Neglect the small diameter of the roller.

Solution: Configuration of the mechanism leads us to adopt the r-θ coordinate frame in describing the motion of roller A. With the reference point O and the reference direction the horizontal line, we formulate the kinematical constraint from the hidden triangle depicted in fig. 8.46. By the cosine law, we

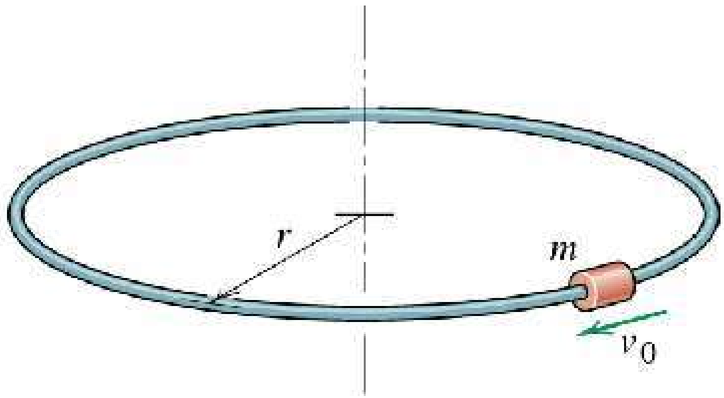


Figure 8.43: Example 8.21 ([3], pp. 156)

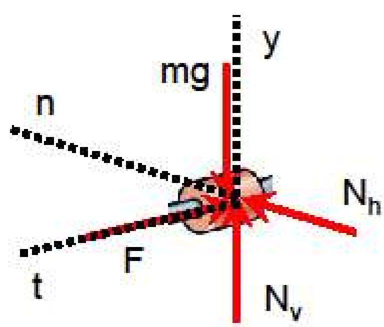


Figure 8.44: Solution to example 8.21

have

$$0.1^2 + r^2 + 0.2r\cos\theta = 0.2^2$$

Differentiate the equation to derive the velocity and acceleration constraints:

$$2r\dot{r} + 0.2\dot{r}\cos\theta - 0.2r\dot{\theta}\sin\theta = 0$$

$$2\dot{r}^2 + 2r\ddot{r} + 0.2\ddot{r}\cos\theta - 0.4\dot{r}\dot{\theta}\sin\theta - 0.2r\ddot{\theta}\sin\theta - 0.2r\dot{\theta}^2\cos\theta = 0$$

With the given motion of the circular cam, $\theta = \frac{\pi}{4}$, $\dot{\theta} = 15$ rad/s, and $\ddot{\theta} = 0$. By the constraints, we may solve for the remaining parameters.

$$r = 0.1164 \text{ m}, \quad \dot{r} = 0.66 \text{ m/s}, \quad \ddot{r} = 15.05 \text{ m/s}^2$$

Kinetics analysis starts from the free body diagram showing all forces acting on the roller. The angle the normal force made with the r-axis, β, may be determined by applying the law of sine to the hidden triangle:

$$\frac{0.2}{\sin 135} = \frac{0.1}{\sin\beta}, \quad \beta = 20.7^\circ$$

Spring force may be determined alternatively from the compressed length. If the distance from O to the roller is r, at any instant the spring is compressed by $r - 0.1$. The compressive force F then becomes

$$F = 5000 \times (r - 0.1)$$

With all above information, the equations of motion may be formulated to solve for the reaction forces, N and R;

$[\Sigma F_r = ma_r]$ $$-F + N\cos 20.7 = m(\ddot{r} - r\dot{\theta}^2)$$

$$-5000 \times (0.1164 - 0.1) + N\cos 20.7 = 0.5(15.05 - 0.1164 \times 15^2), \quad N = 81.7 \text{ N}$$

$[\Sigma F_\theta = ma_\theta]$ $$R - N\sin 20.7 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$R - 81.7\sin 20.7 = 0.5(0.1164 \times 0 + 2 \times 0.66 \times 15), \quad R = 38.7 \text{ N}$$

Example 8.23 ([4], Prob. 3/102) The small cart is nudged with negligible velocity from its horizontal position at A onto the parabolic path that lies in a vertical plane. Neglect friction and show that the cart maintains contact with the path for all values of k.

Solution: To show that the cart maintains contact with the path is equivalent to show that the reaction force (normal force solely in this case) must be greater than zero. This inspires us to use the n-t coordinate frame since the normal force

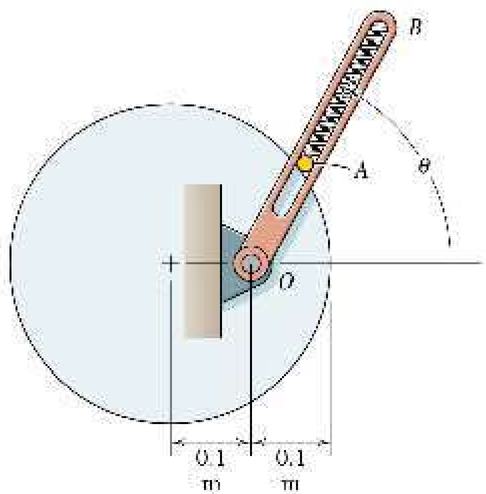


Figure 8.45: Example 8.22 ([4], pp. 156)

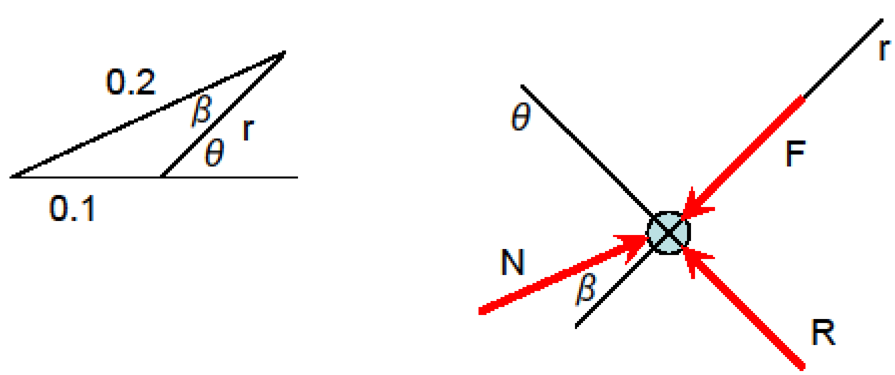


Figure 8.46: Solution to example 8.22

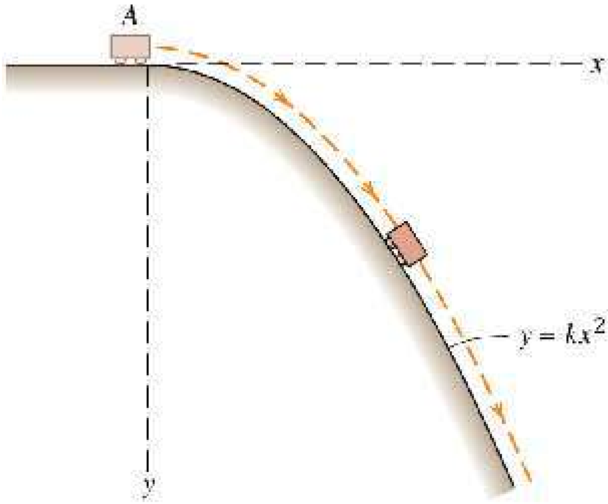


Figure 8.47: Example 8.23 ([4], pp. 156)

N will align with the n-axis at all time. Drawing the free body diagram of the cart is shown in fig. 8.48 where the only forces acting upon it is the gravity and the normal force. Formulation of its equations of motion follws straightforwardly;

$[\Sigma F_n = ma_n] \qquad -N + mg\cos\theta = m\frac{v^2}{\rho}$

$[\Sigma F_t = ma_t] \qquad mg\sin\theta = ma_t$

Radius of curvature, ρ, may be written in terms of the path parameters. The

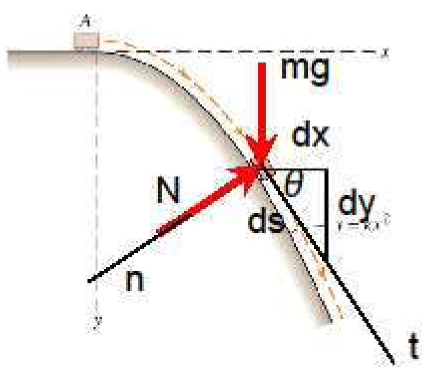


Figure 8.48: Solution to example 8.23

parabolic path is $y = kx^2$. Its derivatives may be evaluated as

$$\frac{dy}{dx} = \tan\theta = 2kx, \quad \frac{d^2y}{dx^2} = 2k$$

Hence by the differential calculus,

$\left[\rho = \frac{\left[1+y'^2\right]^{3/2}}{y''}\right]$ $\qquad\qquad \rho = \frac{\left(1+4k^2x^2\right)^{3/2}}{2k}$

According to the n-t frame notation, t-axis is coincident with the tangent of the path. Hence, the angle θ that it made with the horizontal reference direction may be determined as

$$\theta = \tan^{-1}\left(\frac{dy}{dx}\right)$$

Nevertheless, $\cos\theta$ may be evaluated directly if one recognize the trigonometry identity:

$[1 + \tan^2\theta = \sec^2\theta]$ $\qquad\qquad \cos\theta = 1/\sqrt{1 + 4k^2x^2}$

Because the tangential acceleration is known as a function of the path parameter (i.e. $a_t = g\sin\theta$), the velocity may be determined from

$[vdv = a_t ds]$ $\qquad\qquad vdv = g\sin\theta ds = gdy$

with $dy = ds\sin\theta$ (see differential triangle in fig. 8.48). Total derivative of the above equation yields the required velocity square;

$$v^2 = 2gy = 2kgx^2$$

Substitute the above expressions into the n-component equation of motion, the normal force is

$$N = \frac{mg}{\sqrt{1 + 4k^2x^2}} - 2mkgx^2 \times \frac{2k}{\left(1 + 4k^2x^2\right)^{3/2}}$$

$$N = \frac{mg}{\left(1 + 4k^2x^2\right)^{3/2}} > 0$$

which justify the cart will not get off the unilateral-constraint parabolic path.

Chapter 9

Plane Kinematics of Rigid Bodies

9.1 Introduction

This chapter studies the plane kinematics of rigid bodies. As usual, we will not consider how the motion is achieved in kinematics. This is the subject of kinetics. However rigid bodies motion differ from the particles motion in that the angular motion must be taken care of properly. Typical plane motion of rigid bodies are shown in fig. 9.1. Knowledge of the rigid bodies motion are important in designing a mechanism to perform the desired motion. Kinematics is the prerequisite to Kinetics. Together, they will help us determine the motion resulting from the applied force. Now some important notions will be reviewed before the concrete analysis.

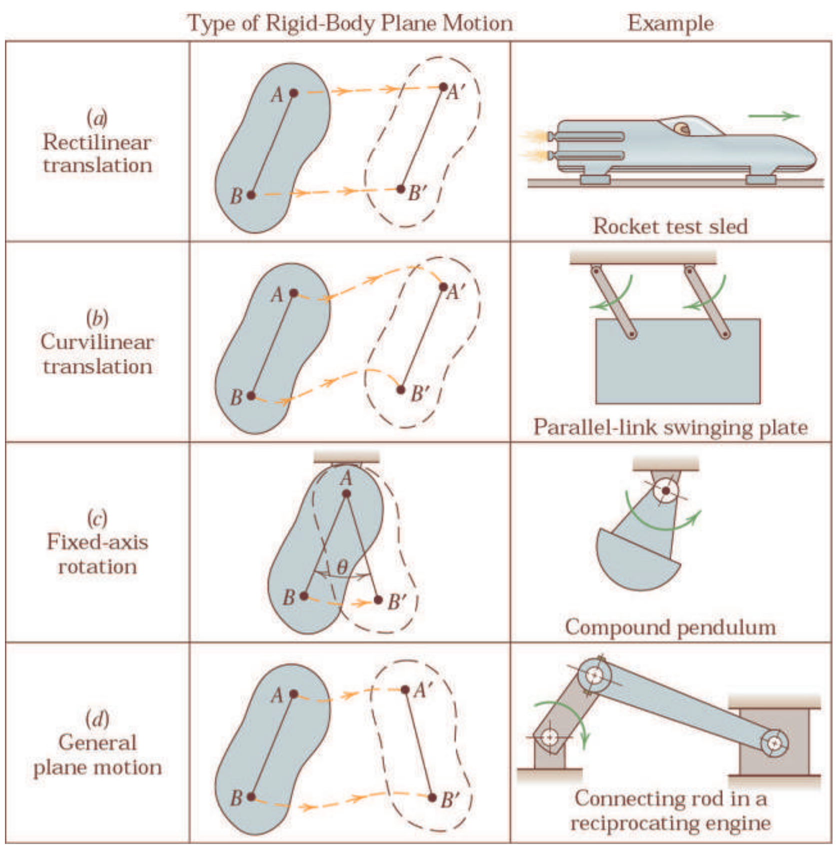


Figure 9.1: Common plane motion of rigid bodies ([3], pp. 333)

Rigid body is a system of particles for which the distance between the particles remain unchanged. Thus there will be *no change in the position vector of any particle measured from the body-fixed coordinate system.*

Plane motion of a rigid body is a specific motion in which all parts of the body move in parallel planes. Hence the body can be treated as a thin slab with motion confined to the *plane of motion*; the plane that contains the center of mass.

Translation is the motion in which every *line* in the body remains parallel to its original position at all time. In other words, there is *no rotation of any line in the body*. The motion of the body is completely specified by the motion of any *point* in the body, since all points have the same motion.

Rectilinear translation All points in the body move in parallel straight lines of the same distance.

Curvilinear translation All points move on parallel curves of the same distance.

Rotation is the motion in which all particles move in circular paths about the axis of rotation. *All lines in the body which are perpendicular to the axis of rotation rotate through the same angle in the same time*. Therefore circular motion of a point helps describe the rotating motion.

General plane motion is the combination of translation and rotation. See case (d) of fig. 9.1. The principle of relative motion helps describe the general motion.

There are two main approaches for the kinematics analysis. The first approach employs the geometry of the problem at hand for direct calculation of the absolute displacement, velocity, and acceleration. The second method, which is easier for most of the time, use the principle of relative motion. Later sections contain details and examples of both methods.

9.2 Rotation

Rotation of a rigid body is described by its angular motion, which is dictated by the *change* in the angular position (specified by the angle θ measured from *any* fixed line) of any line attached to the body. Figure 9.2 shows the rotation motion of a rigid body. θ_1 and θ_2 are the angles of line 1 and 2 measured from any fixed reference line.

Their difference is the angle β of which its value does not matter. Its consequences can be written as

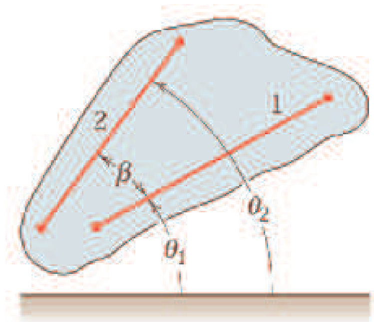


Figure 9.2: Rotation motion of a rigid body ([3], pp. 333)

$$\theta_2 = \theta_1 + \beta$$
$$\Delta\theta_2 = \Delta\theta_1$$
$$\dot{\theta}_2 = \dot{\theta}_1$$
$$\ddot{\theta}_2 = \ddot{\theta}_1$$

9.2.1 Angular Motion Relations

The above mathematics implies that all lines on a rigid body in its plane of motion have the same angular displacement, the same angular velocity, and the same angular acceleration. Therefore, we can pick up any line in the plane of motion and associate it with the angular position coordinate θ. The angular velocity ω and acceleration α of a rigid body in plane rotation are defined as

$$\omega = \dot{\theta} \tag{9.1}$$

$$\alpha = \dot{\omega} = \ddot{\theta} \tag{9.2}$$

$$\omega\, d\omega = \alpha\, d\theta \quad \text{or} \quad \dot{\theta}\, d\dot{\theta} = \ddot{\theta}\, d\theta \tag{9.3}$$

Note the analogies between the linear and angular motion.

9.2.2 Rotation about a Fixed Axis

If a rigid body rotate in a plane about a fixed perpendicular axis, all points other than those on the rotation axis will move in concentric circles about the fixed axis. Hence, the curvilinear motion of a point A is related to the angular motion of the rigid body by the familiar n-t coordinate frame kinematic relationship. Refer to fig. 9.3 and fig. 9.4. In scalar form, motion of A can be written as

$$v = r\omega, \quad r = \text{constant} \tag{9.4}$$

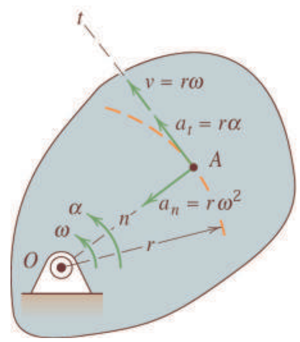


Figure 9.3: Rotation about a fixed axis viewing from top ([3], pp. 335)

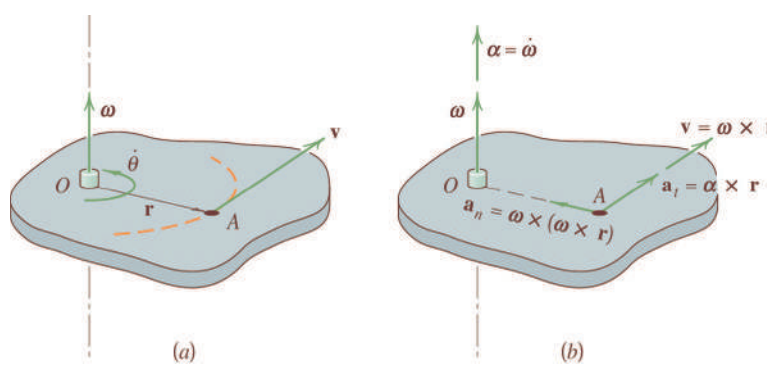


Figure 9.4: Rotation about a fixed axis in perspective view ([3], pp. 335)

$$a_{\mathrm{t}} = \dot{v} = r\alpha \tag{9.5}$$

$$a_{\mathrm{n}} = r\omega^2 = \frac{v^2}{r} \tag{9.6}$$

In vector form (fig. 9.4), if ω is the angular velocity of the position vector $\mathbf{r}$ (constant magnitude) which is also the angular velocity of the rigid body, motion of A can be written as

$$\mathbf{v} = \dot{\mathbf{r}} = \omega \times \mathbf{r} \tag{9.7}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \omega \times (\omega \times \mathbf{r}) + \alpha \times \mathbf{r} \tag{9.8}$$

$$\mathbf{a}_{\mathrm{n}} = \omega \times (\omega \times \mathbf{r}) \tag{9.9}$$

$$\mathbf{a}_{\mathrm{t}} = \alpha \times \mathbf{r} \tag{9.10}$$

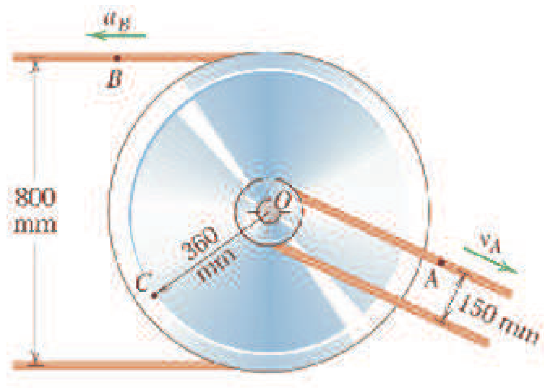


Figure 9.5: Example 9.1 ([3], pp. 342)

Example 9.1 ([3], Prob. 5/22) The two V-belt pulleys form an integral unit and rotate about the fixed axis at O. At a certain instant, point A on the belt of the smaller pulley has a velocity $v_A = 1.5$ m/s, and point B on the belt of the larger pulley has an acceleration $a_B = 45$ m/s^2 as shown. For this instant, determine the magnitude of the acceleration a_C of point C and sketch the vector in your solution.

Solution: To determine the acceleration, the velocity must first be determined. Since point C on the pulley is moving in circular motion, the pulley angular velocity and acceleration must be known. These can be calculated from the motion of the belts with the assumption that there is no slippage between the belt and the V-groove. First, find ω_{pulley} from v_A.

$[v = \omega r]$ $$1.5 = \omega \times 0.075, \quad \omega = 20 \text{ rad/s CW}$$

The angular acceleration of the pulley is determined from a_B.

$[a_t = r\alpha]$ $$45 = 0.4 \times \alpha, \quad \alpha = 112.5 \text{ rad/s}^2 \text{ CCW}$$

Now we are ready to determine the acceleration at C.

$[a_n = r\omega^2]$ $$a_{C_n} = 0.36 \times 20^2 = 144 \text{ m/s}^2$$

$[a_t = r\alpha]$ $$a_{C_t} = 0.36 \times 112.5 = 40.5 \text{ m/s}^2$$

$$a_C = \sqrt{a_{C_n}^2 + a_{C_t}^2} = 149.6 \text{ m/s}^2$$

Example 9.2 ([3], Prob. 5/26) A V-belt speed-reduction drive is shown where pulley A drives the two integral pulleys B which in turn drive pulley C. If

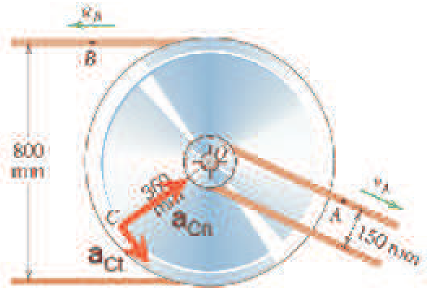


Figure 9.6: Solution to example 9.1

A starts from rest at time $t = 0$ and is given a constant angular acceleration α_1, derive expressions for the angular velocity of C and the magnitude of the acceleration of a point P on the belt, both at time t.

Solution: Since the angular acceleration of pulley A is constant and the pulley starts from rest, the angular velocity at time t can be determined easily:

$\left[\alpha = \frac{d\omega}{dt}\right]$ $$\omega_A = \alpha_1 t$$

Assume the pulleys roll without slipping with the belt and the belt's elasticity is negligible. By these kinematic constraints, the angular velocity and acceleration of pulley B and C are

$$\omega_B = \left(\frac{r_1}{r_2}\right)\alpha_1 t, \quad \omega_C = \left(\frac{r_1}{r_2}\right)^2 \alpha_1 t$$

$$\alpha_B = \left(\frac{r_1}{r_2}\right)\alpha_1, \quad \alpha_C = \left(\frac{r_1}{r_2}\right)^2 \alpha_1$$

Point P moves in circular path with the center of the curvature at the center point of pulley C. Therefore its acceleration comprises of the normal and tangential acceleration, of which the formulas are well familiar.

$[a_n = r\omega^2]$ $$a_{P_n} = r_2\left(\frac{r_1}{r_2}\right)^4 (\alpha_1 t)^2$$

$[a_t = r\alpha]$ $$a_{P_t} = r_2\left(\frac{r_1}{r_2}\right)^2 \alpha_1$$

The magnitude of the acceleration of P is hence

$$a_P = \sqrt{a_{P_n}^2 + a_{P_t}^2} = \frac{r_1^2}{r_2}\alpha_1\sqrt{1 + \left(\frac{r_1}{r_2}\right)^4 \alpha_1^2 t^4}$$

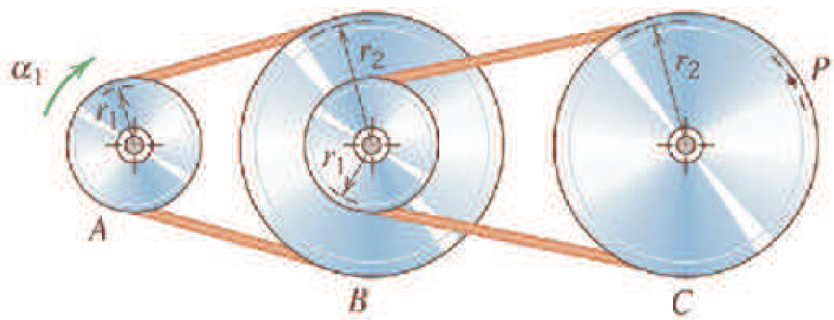


Figure 9.7: Example 9.2 ([3], pp. 343)

9.3 Absolute Motion

An approach to kinematic analysis is the absolute motion method. The process is very straightforward. It starts with determining the geometric relations that define the configuration involved. Then, the time derivatives of the relations are perfomed to obtain velocities and accelerations. The sign consistency must be kept throughout the analysis.

The crucial step, also the most difficult one, is to determine the geometric configuration. Consequently, some problems with complicated geometries are not suitable to analyze with this method as the constraints and the mathematics become increasingly involved. Instead, the principle of relative motion, introduced in section 9.4, is recommended.

Example 9.3 ([3], SP. 5/4) A wheel of radius r rolls on a flat surface without slipping. Determine the angular motion of the wheel in terms of the linear motion of its center O. Also determine the acceleration of a point on the rim of the wheel as the point comes into contact with the surface on which the wheel rolls.

Solution: The problem states the rolling without slipping condition of the wheel motion. This is in fact important which helps identifying the kinematic relationship. In the absolute motion approach, the kinematic relationship must first be determined. For this specific problem with the given condition, it can be concluded that *the displacement of the center O must be equal to the arc length along the rim of the wheel that rolls over the flat surface*. With the depicted figure, fig. 9.8, the distance s is equal to the arc length $C'A$. Mathematically,

$$s = r\theta$$

Differentiating the above relationship, we have the velocity and the acceleration relations;

$$v_O = r\omega$$

$$a_O = r\alpha$$

where

v_O	=	velocity of the *center* O of the wheel
ω	=	angular velocity of the wheel
a_O	=	acceleration of the *center* O of the wheel
α	=	angular acceleration of the wheel

The above equations are very similar to the equations related to the rotation about the fixed axis. This is the rolling without slipping motion. *Do not* screw up!

To determine the instantaneous acceleration of the contact point of the wheel and the surface, set up a fixed coordinate system originated at the point of contact. Observer in this frame will see point C moving along its absolute path; *the cycloidal path*. From the figure, the x-y coordinates of point C at the current location C' is

$$x = s - r\sin\theta = r(\theta - \sin\theta)$$

$$y = r - r\cos\theta = r(1 - \cos\theta)$$

Take derivatives with respect to time and apply the previous relationship,

$$\dot{x} = r\dot{\theta}(1 - \cos\theta) = v_O(1 - \cos\theta)$$

$$\dot{y} = r\dot{\theta}\sin\theta = v_O\sin\theta$$

$$\ddot{x} = \dot{v}_O(1 - \cos\theta) + v_O\dot{\theta}\sin\theta = a_O(1 - \cos\theta) + r\omega^2\sin\theta$$

$$\ddot{y} = \dot{v}_O\sin\theta + v_O\dot{\theta}\cos\theta = a_O\sin\theta + r\omega^2\cos\theta$$

When point C comes to contact, $\theta = 0$. Substitute this value into the above equations.

$$\dot{x} = 0 \qquad \dot{y} = 0$$

$$\ddot{x} = 0 \qquad \ddot{y} = r\omega^2$$

Point C has zero velocity as expected from the constraint of rolling without slipping. However it has the acceleration of $r\omega^2$ pointing normal off the surface. This is in fact the tangential acceleration along the cycloidal path to take it from rest.

There are frequent situations for which the rolling without slipping motion can be assumed. See fig. 9.9. There we can apply these basic relations immediately.

Example 9.4 ([3], SP. 5/5) The load L is being hoisted by the pulley and cable arrangement shown. Each cable is wrapped securely around its respective pulley so it does not slip. The two pulleys to which L is attached are fastened together to form a single rigid body. Calculate the velocity and acceleration of the load L and the corresponding angular velocity ω and angular acceleration α of the

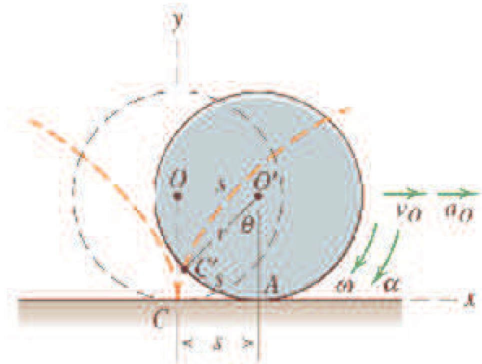


Figure 9.8: Example 9.3 ([3], pp. 345)

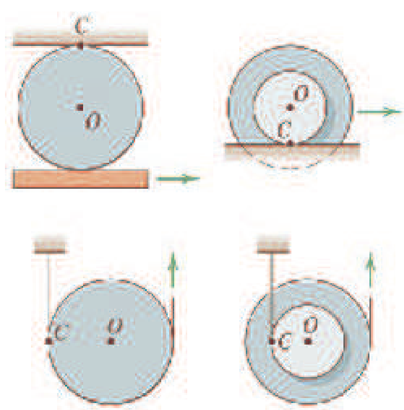


Figure 9.9: Applications of rolling without slipping ([3], pp. 345)

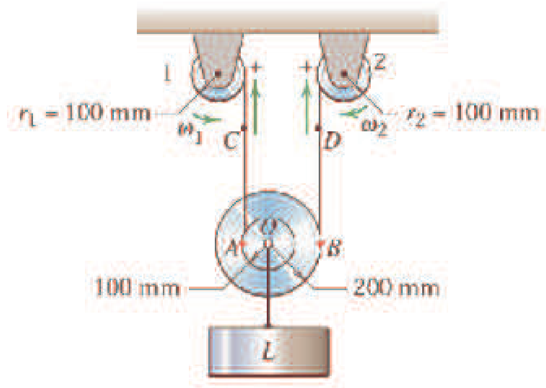


Figure 9.10: Example 9.4 ([3], pp. 346)

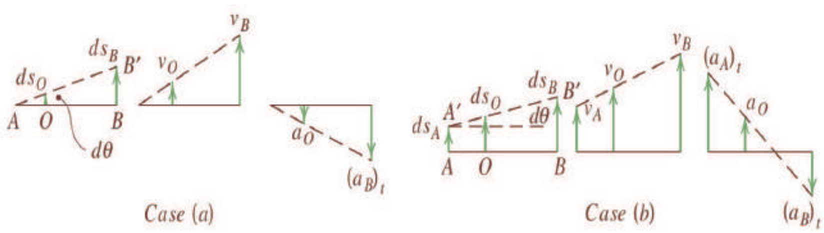


Figure 9.11: Solution to example 9.4 ([3], pp. 346)

double pulley under the following conditions:

Case (a):	Pulley 1:	$\omega_1 = 0, \quad \alpha_1 = 0$ (pulley at rest)
	Pulley 2:	$\omega_2 = 2 \ \ \text{rad/s}, \quad \alpha_2 = -3 \ \ \text{rad/s}^2$
Case (b):	Pulley 1:	$\omega_1 = 1 \ \ \text{rad/s}, \quad \alpha_1 = 4 \ \ \text{rad/s}^2$
	Pulley 2:	$\omega_2 = 2 \ \ \text{rad/s}, \quad \alpha_2 = -2 \ \ \text{rad/s}^2$

Solution:

Example 9.5 ([3], Prob. 5/35) The telephone-cable reel rolls without slipping on the horizontal surface. If point A on the cable has a velocity $v_A = 0.8$m/s to the right, compute the velocity of the center O and the angular velocity ω of the reel. (Be careful not to make the mistake of assuming that the reel rolls to the left.)

Solution: From the previous analysis, rolling without slipping implies the velocity at the contact point is zero. Also there is no slippage of the cable at the inner hub, which implies the velocity of the contact rim is the same as the velocity of the wrapped cable. Observe the motion of any line on the reel,

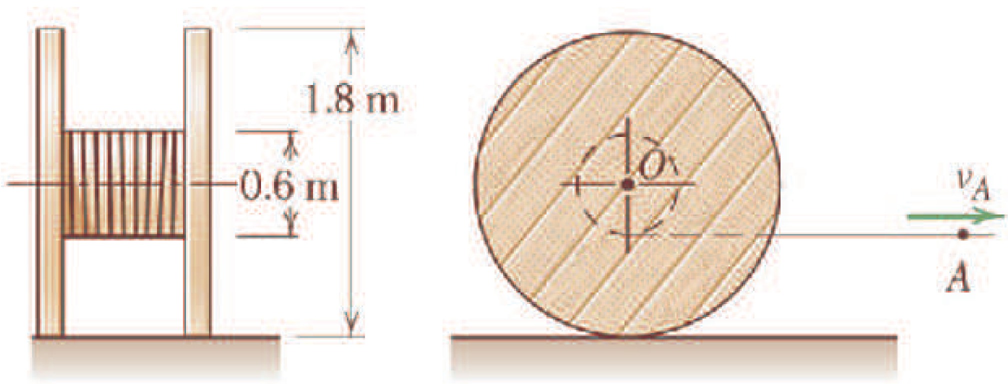


Figure 9.12: Example 9.5 ([3], pp. 350)

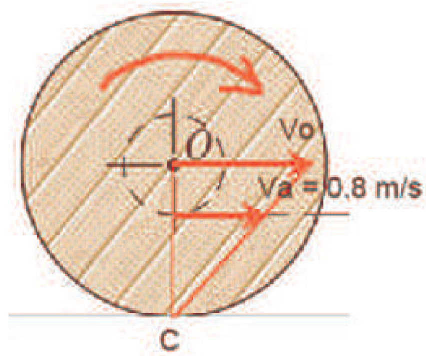


Figure 9.13: Solution to example 9.5

here line OC, to determine the angular motion. From the proportional velocity profile on the disc, fig. 9.13,

$$\frac{v_O}{0.9} = \frac{0.8}{0.6}, \quad v_O = 1.2 \ \text{m/s} \quad \rightarrow$$

$[v_O = \omega r]$ $$\omega = \tfrac{1.2}{0.9} = 1.333 \ \text{rad/s} \ \ \text{CW}$$

Example 9.6 ([4], Prob. 5/37) The cable from the drum A turns the double wheel B, which rolls on its hubs without slipping. Determine the angular velocity ω and angular acceleration α of drum C for the instant when the angular velocity and angular acceleration of A are 4rad/s and 3rad/s^2, respectively, both in the CCW direction.

Solution: Drum A imparts its motion to the double wheel B through the wrapped cable. In the same manner, drum C rolls CCW by the CW rotation of the outer wheel B through the wrapped cable. Assume there is no slip of the wrapped cable, which makes its motion the same as the tangential component of

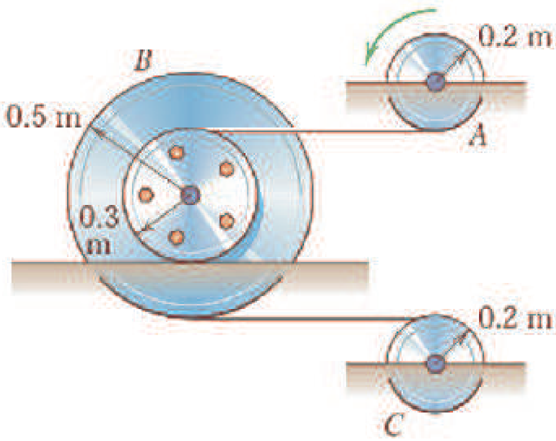


Figure 9.14: Example 9.6 ([4], pp. 340)

the departed point on the drum. Therefore the motion of the upper cable is

$[v = r\omega]$ $$v_A = 0.2 \times 4 = 0.8 \ \ \text{m/s} \quad \rightarrow$$

$[a_t = r\alpha]$ $$(a_A)_t = 0.2 \times 3 = 0.6 \ \ \text{m/s}^2 \quad \rightarrow$$

Wheel B rolls with the induced motion of the connected cable:

$[\omega = v/r]$ $$\omega_B = 0.8/0.6 = 4/3 \ \ \text{rad/s} \ \ \text{CW}$$

$[\alpha = a_t/r]$ $$\alpha_B = 0.6/0.6 = 1 \ \ \text{rad/s}^2 \ \ \text{CW}$$

The lower cable has the same motion as the tangential component of the departed point from wheel B as

$[v = r\omega]$ $$v_C = 4/3 \times 0.2 = 0.267 \ \ \text{m/s} \quad \leftarrow$$

$[a_t = r\alpha]$ $$(a_C)_t = 1 \times 0.2 = 0.2 \ \ \text{m/s}^2 \quad \leftarrow$$

Figure 9.15 displays the velocity profile distribution of the double wheel B. Finally, drum C rolls by the motion of the tensioning lower cable:

$[\omega = v/r]$ $$\omega_C = 0.267/0.2 = 4/3 \ \ \text{rad/s} \ \ \text{CCW}$$

$[\alpha = a_t/r]$ $$\alpha_C = 0.2/0.2 = 1 \ \ \text{rad/s}^2 \ \ \text{CCW}$$

Example 9.7 ([4], Prob. 5/45) The rod OB slides through the collar pivoted to the rotating link at A. If CA has an angular velocity $\omega = 3\text{rad/s}$ for an interval

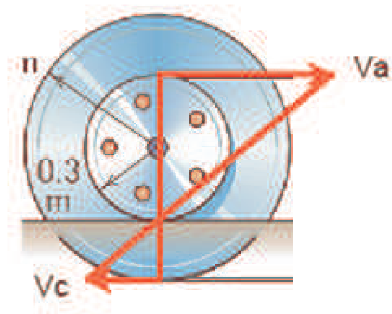


Figure 9.15: Solution to example 9.6

of motion, calculate the angular velocity of OB when $\theta = 45°$.

Solution: The constraint geometry of this problem is rather obvious that OAC must form a triangle. One can then use the law of sine and cosine relationship and their derivatives to determine the missing motion. However, the other relationships for this particular mechanism is that the vertical projection of the link CA and OA must add up to the fixed length of 400 mm. Also the horizontal projection of both links must be equal for the links to join at A. Mathematically,

$$\overline{CA}\cos\theta + y\cos\beta = 0.4$$

$$y\sin\beta - \overline{CA}\sin\theta = 0$$

With the given parameters $\theta = 45°$ and $\overline{CA} = 0.2$ m, we can solve for y and β from the above equations:

$$y = 0.2947 \text{ m} \quad \text{and} \quad \beta = 28.675°$$

Differentiate the geometric relations with respect to time, we have

$$-\overline{CA}\dot{\theta}\sin\theta + \dot{y}\cos\beta - y\dot{\beta}\sin\beta = 0$$

$$\dot{y}\sin\beta + y\dot{\beta}\cos\beta - \overline{CA}\dot{\theta}\cos\theta = 0$$

Given $\dot{\theta} = -3$ rad/s, we can solve for the angular velocity of OB, $\dot{\beta}$, from the above equations:

$$\dot{y} = -0.576 \text{ m/s} \quad \text{and} \quad \dot{\beta} = -0.572 \text{ rad/s}$$

Example 9.8 ([3], Prob. 5/54) Show that the expressions $v = r\omega$ and $a_t = r\alpha$ hold for the motion of the center O of the wheel which rolls on the concave or convex circular arc, where ω and α are the absolute angular velocity and acceleration, respectively, of the wheel. (*Hint*: Follow the sample problem and

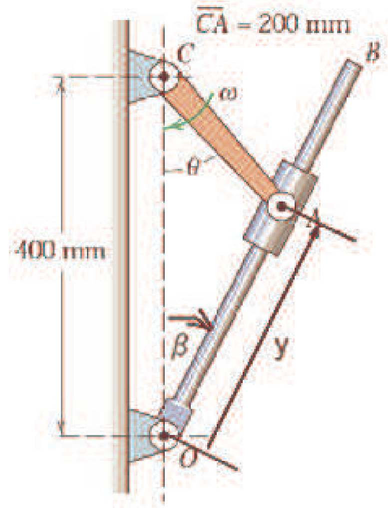


Figure 9.16: Example 9.7 ([4], pp. 342)

allow the wheel to roll a small distance. Be very careful to identify the correct absolute angle through wihch the wheel turns in each case in determining its angular velocity and angular acceleration.)

Solution: Let θ be the angular motion of the wheel while β be the angular motion of the line connecting the wheel center and the center of the curvature arc as shown in fig. 9.18. Consider each case separately.

Concave arc:
The rolling distance can be calculated from either the wheel or the constraining arc, which must be equal. This is the underlying geometrical relationship.

$$\text{rolling distance} = R\beta = r(\theta + \beta)$$

From the figure, the traveling distance of the center O is

$$s = (R - r)\beta$$

Referring to the geometrical relationship, the traveling distance can be written as the function of the wheel motion as

$$s = (R - r)\beta = r\theta$$

Differentiate with respect to time to get the velocity and the tangential acceleration of O:

$$v = \dot{s} = (R - r)\dot{\beta} = r\dot{\theta} = r\omega$$
$$a_t = \dot{v} = (R - r)\ddot{\beta} = r\dot{\omega} = r\alpha$$

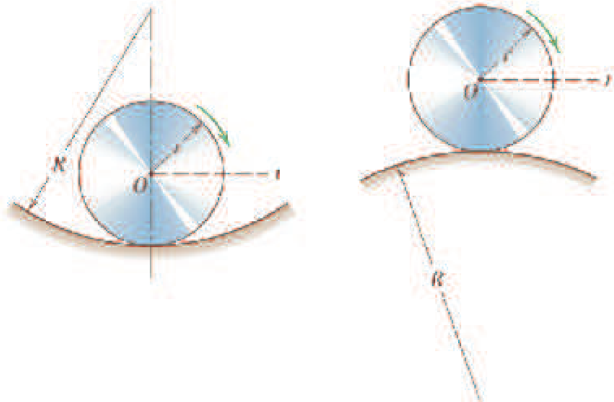


Figure 9.17: Example 9.8 ([3], pp. 354)

Convex arc:
The rolling distance can be calculated from either the wheel or the constraining arc, which must be equal. This is the underlying geometrical relationship.

$$\text{rolling distance} = R\beta = r\,(\theta - \beta)$$

From the figure, the traveling distance of the center O is

$$s = (R + r)\,\beta$$

Referring to the geometrical relationship, the traveling distance can be written as the function of the wheel motion as

$$s = (R + r)\,\beta = r\theta$$

Differentiate with respect to time to get the velocity and the tangential acceleration of O:

$$v = \dot{s} = (R + r)\,\dot{\beta} = r\dot{\theta} = r\omega$$
$$a_t = \dot{v} = (R + r)\,\ddot{\beta} = r\dot{\omega} = r\alpha$$

Therefore, it can be conclude that the velocity and the tangential acceleration of the center of the wheel, O,

$$v = r\omega$$

$$a_t = r\alpha$$

hold independent of the curvature, R, of the terrain!

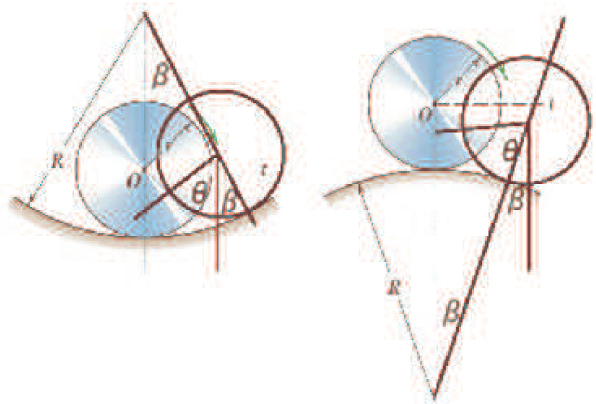


Figure 9.18: Solution to example 9.8

Example 9.9 ([3], Prob. 5/56) The Geneva wheel is a mechanism for producing intermittent rotation. Pin P in the integral unit of wheel A and locking plate B engages the radial slots in wheel C thus turning wheel C one-fourth of a revolution for each revolution of the pin. At the engagement postion shown, $\theta = 45°$. For a constant CW angular velocity $\omega_1 = 2$rad/s of wheel A, determine the corresponding CCW angular velocity ω_2 of the wheel C for $\theta = 20°$. (Note that the motion during engagement is governed by the geometry of triangle O_1O_2P with changing θ.)

Solution: The governing geometry of this mechanism is the triangle O_1O_2P. According to fig. 9.19, $\dot{\beta}$ is the angular velocity of wheel C. Therefore the appropriate relation must contain the parameter β so differentiating the equation will give the desired angular velocity of the wheel C. The kinematical relationship is

$$\tan\beta = \frac{\overline{O_1P}\sin\theta}{\overline{O_1O_2} - \overline{O_1P}\cos\theta} = \frac{\frac{1}{\sqrt{2}}\sin\theta}{1 - \frac{1}{\sqrt{2}}\cos\theta}$$

Differentiating the equation gives,

$$\dot{\beta}\sec^2\beta = \frac{\left(1 - \frac{1}{\sqrt{2}}\cos\theta\right) \times \frac{1}{\sqrt{2}}\dot{\theta}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta \times \left(\frac{1}{\sqrt{2}}\dot{\theta}\sin\theta\right)}{\left(1 - \frac{1}{\sqrt{2}}\cos\theta\right)^2}$$

Substituting the given motion of wheel A,

$$\theta = 20°, \quad \dot{\theta} = -2 \ \text{rad/s}, \quad \ddot{\theta} = 0$$

in the above equations and solving for the unknowns result in

$$\beta = 35.783°, \quad \dot{\beta} = -1.923 \ \text{rad/s}$$

Hence the angular velocity of the wheel C is 1.923 rad/s CCW.

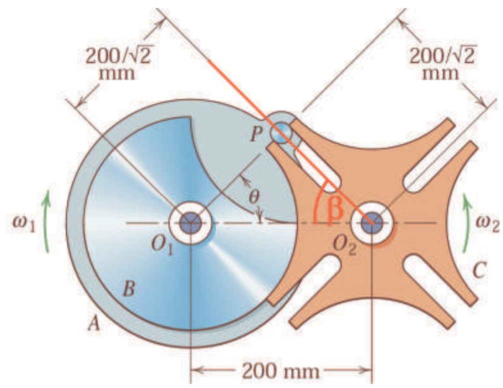


Figure 9.19: Example 9.9 ([3], pp. 355)

Example 9.10 ([4], Prob. 5/54) The rod AB slides through the pivoted collar as end A moves along the slot. If A starts from rest at $x = 0$ and moves to the right with a constant acceleration of 0.1 m/s^2, calculate the angular acceleration α of AB at the instant when $x = 150$mm.

Solution: The angular acceleration of AB is $\ddot{\theta}$. Therefore the underlying geometry must contain the angle θ so differentiation will lead to the desired angular acceleration. An obvious one is the right triangle. The relationship is

$$x = 0.2 \tan\theta$$

Differentiating twice gives

$$\dot{x} = 0.2\dot{\theta}\sec^2\theta$$
$$\ddot{x} = 0.2\ddot{\theta}\sec^2\theta + 0.2\dot{\theta} \times \left(2\dot{\theta}\sec^2\theta\tan\theta\right)$$

A starts from rest at $x = 0$ and moves to the right with a constant acceleration of 0.1 m/s^2. This implies

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad \ddot{x} = 0.1 \ \text{ m/s}^2 \ \text{ constant}$$

Integrating to determine the velocity and motion relation;

$$\dot{x} = 0.1t, \quad x = 0.05t^2$$

At $x = 0.15$ m, $t = \sqrt{3}$ sec. At that instant, $\dot{x} = 0.1\sqrt{3}$ m/s and $\ddot{x} = 0.1$ m/s^2. At that posture, $\tan\theta = 3/4$ and $\sec\theta = 5/4$. Substituting these parameters into the above equations, the angular motion of AB are

$$\dot{\theta} = 0.554 \ \text{ rad/s}, \quad \ddot{\theta} = -0.1408 \ \text{ rad/s}^2$$

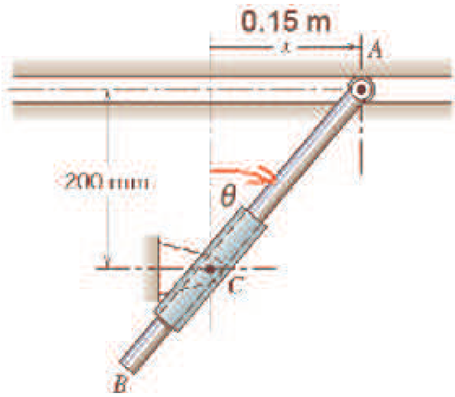


Figure 9.20: Example 9.10 ([4], pp. 345)

Example 9.11 ([3], Prob. 5/57) The punch is operated by a simple harmonic oscillation of the pivoted sector given by $\theta = \theta_0 \sin 2\pi t$ where the amplitude is $\theta_0 = \frac{\pi}{12}\text{rad}\,(15°)$ and the time for one complete oscillation is 1 second. Determine the acceleration of the punch when (a) $\theta = 0$ and (b) $\theta = \frac{\pi}{12}$.

Solution: The constrained geometry inside this mechanism is the triangle above the punch. Apply the law of cosine to relate the angular to the translational displacement as follow;

$$0.1^2 = y^2 + 0.14^2 - 0.28y\cos\theta$$

Differentiate twice to get the acceleration relationship.

$$0 = 2y\dot{y} - 0.28\dot{y}\cos\theta + 0.28y\dot{\theta}\sin\theta$$

$$0 = 2\dot{y}^2 + 2y\ddot{y} - 0.28\ddot{y}\cos\theta + 0.28\dot{y}\dot{\theta}\sin\theta + 0.28\dot{y}\dot{\theta}\sin\theta + 0.28y\ddot{\theta}\sin\theta + 0.28y\dot{\theta}^2\cos\theta$$

Motion of the pivoted sector is a simple harmonic oscillation described by

$$\theta = \theta_o \sin 2\pi t$$

Differentiate twice to get the angular acceleration, $\ddot{\theta}$;

$$\dot{\theta} = 2\pi\theta_o \cos 2\pi t$$

$$\ddot{\theta} = -\left(2\pi\right)^2 \theta_o \sin 2\pi t$$

When $\theta = 0, \quad t = 0$ s. Consequently,

$$\dot{\theta} = 2\pi\theta_o, \quad \ddot{\theta} = 0$$

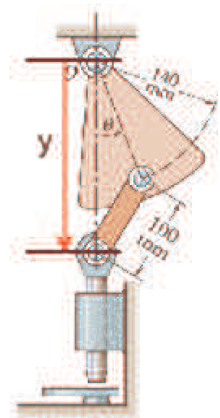


Figure 9.21: Example 9.11 ([3], pp. 355)

Substitute these parameters into the triangular relation, motion of the punch is

$$\begin{aligned} y &= \underline{0.24}, 0.04 \ \text{ m} \\ \dot{y} &= 0 \ \text{ m/s} \\ \ddot{y} &= -0.909 \ \text{ m/s}^2 \end{aligned}$$

When $\theta = \frac{\pi}{12}, \quad t = \frac{1}{4}$ s. Consequently,

$$\dot{\theta} = 0, \quad \ddot{\theta} = -(2\pi)^2 \theta_o$$

Substitute these parameters into the triangular relation, motion of the punch is

$$\begin{aligned} y &= \underline{0.2284}, 0.042 \ \text{ m} \\ \dot{y} &= 0 \ \text{ m/s} \\ \ddot{y} &= 0.918 \ \text{ m/s}^2 \end{aligned}$$

Example 9.12 ([3], Prob. 5/58) One of the most common mechanisms is the slider-crank. Express the angular velocity ω_{AB} and angular acceleration α_{AB} of the connecting rod AB in terms of the crank angle θ for a given constant crank speed ω_0. Take ω_{AB} and α_{AB} to be positive counterclockwise.

Solution: The underlying geometry is inside the triangle ABO. It is clear that the height of B calculating from either AB or OB must be equal. Therefore

$$l \sin\beta = r \sin\theta$$

Differentiate with respect to time to get the velocity relationship:

$$l\dot{\beta}\cos\beta = r\dot{\theta}\cos\theta$$

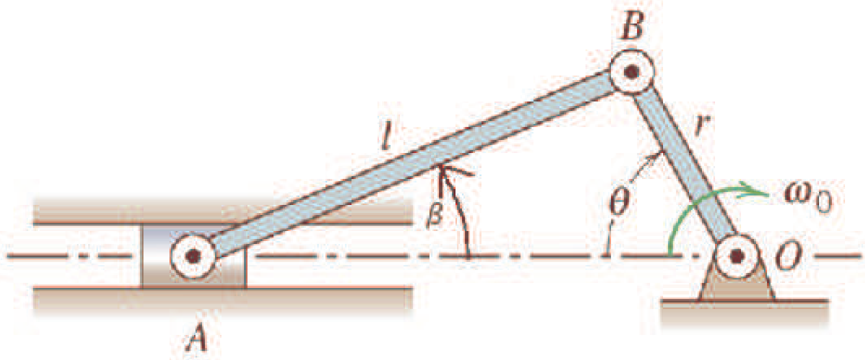


Figure 9.22: Example 9.12 ([3], pp. 355)

and substitute $\dot{\theta} = \omega_0$ so that

$$\omega_{AB} = \dot{\beta} = \frac{r\omega_0 \cos\theta}{l\cos\beta} = \frac{r\omega_0}{l}\frac{\cos\theta}{\sqrt{1-\frac{r^2}{l^2}\sin^2\theta}}$$

Differentiate again to get the acceleration relationship:

$$l\ddot{\beta}\cos\beta - l\dot{\beta}^2\sin\beta = r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta$$

and substitute $\beta, \dot{\beta}, \dot{\theta},$ and $\ddot{\theta} = 0$ so that

$$\alpha_{AB} = \ddot{\beta} = \frac{l\dot{\beta}^2\sin\beta - r\dot{\theta}^2\sin\theta}{l\cos\beta} = \frac{r\omega_0^2}{l}\sin\theta\frac{\frac{r^2}{l^2}-1}{\left(1-\frac{r^2}{l^2}\sin^2\theta\right)^{3/2}}$$

9.4 Relative Velocity

Another method to analyze the kinematics problems is the principle of relative motion. It is usually suitable for the complex motion as it is more scalable and more systematic. Consider first the velocity of points in a rigid body.

Velocity propagation in the rigid body

Referring to chapter 7, the relative velocity equation using the *non-rotating reference frame* is

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \tag{9.11}$$

Let the two points A and B be on the *same rigid body*. The implication of this choice is that *the motion of one point as seen by an observer translating with the other point must be circular* since the radial distance to the observed point from the reference point does not change.

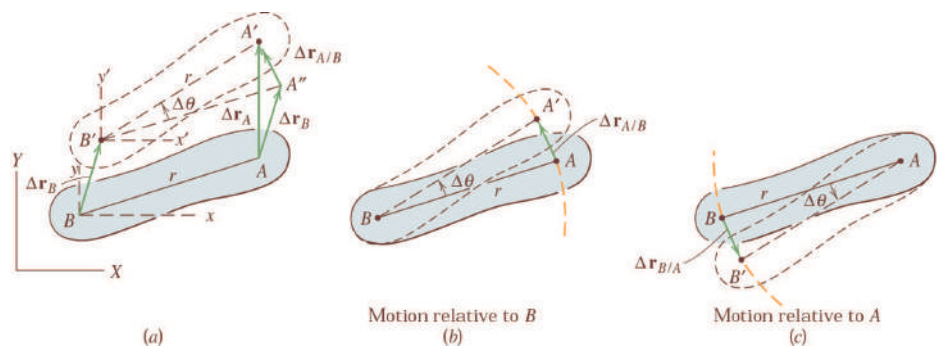


Figure 9.23: General motion of the rigid body: translation and rotation ([3], pp. 356)

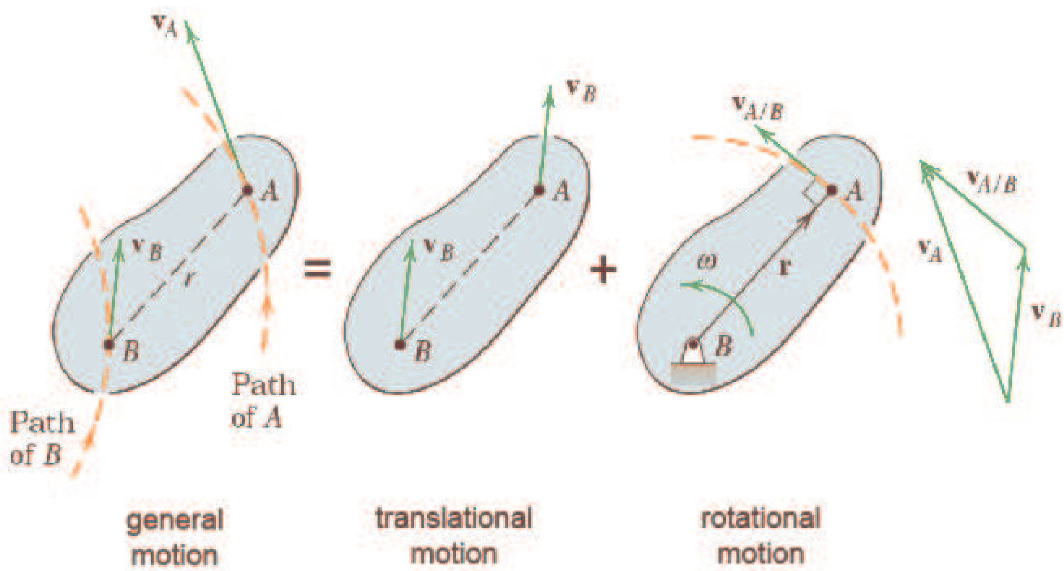


Figure 9.24: General motion of the rigid body: translational and rotational velocity ([3], pp. 357)

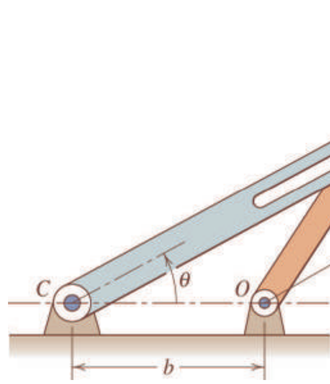


Figure 9.25: Velocity propagation among rigid bodies ([3], pp. 366)

Motion of the rigid body can be divided into two parts: translation and rotation. In fig. 9.23, after the translation of the rigid body expressed by the motion of point B, the body *appears to undergo fixed axis rotation about B with A executing circular motion* as shown in fig. 9.23 (b). Hence the relationship for circular motion describes the relative portion of A's motion.

With B as the reference point, the total displacement of A is

$$\Delta \mathbf{r}_A = \Delta \mathbf{r}_B + \Delta \mathbf{r}_{A/B} \tag{9.12}$$

$\Delta \mathbf{r}_{A/B} = -\Delta \mathbf{r}_{B/A}$ has the magnitude $r\Delta\theta$ as $\Delta\theta \to 0$.

Divide the displacement quantities above by the elapsed time Δt and consider the limit as $\Delta t \to 0$, the familiar relative velocity equation, eq. 9.11 results.

If the distance r between A and B is constant, $\mathbf{v}_{A/B}$ is the velocity of the circular motion. That is

$$\mathbf{v}_{A/B} = \omega \times \mathbf{r}_{A/B} \tag{9.13}$$

or in scalar form

$$v_{A/B} = r_{A/B}\,\omega, \qquad \mathbf{v}_{A/B} \perp \overline{AB} \tag{9.14}$$

where ω is the *absolute* angular velocity of the rigid body. Fig. 9.24 is the analog of fig. 9.23 in the velocity form.

We usually are not interested in just only the motion of points on the same rigid body. Many useful mechanisms are constructed from rigid bodies connected together in a specified manner. Therefore we also would like to determine the motion of points on other bodies from the motion of known points and further information.

Velocity propagation among rigid bodies

Recall the relative velocity equation expressed in the *non-rotating reference* *frame* shown in eq. 9.11. This time, point A and B are *coincident points* on *different rigid bodies* for the instant. See fig. 9.25. Two points are at the position of the pin in the slot. In this case, *the distance between two points are not constrained to be fixed* even it happens to be zero at the moment. Therefore the relative term $\mathbf{v}_{A/B}$ can no more be determined by $\omega \times \mathbf{r}_{A/B}$.

Methods in solving the relative velocity equation

There are several methods in solving eq. 9.11. Here the following three approaches are presented.

1. Vector algebra approach
2. Graphical analysis approach
3. Vector/Graphic approach

Vector algebra approach For this approach, we write each vetor in terms of i- and j- components. Since the problem of interest is planar, there will be two scalar equations for each vector equation. This implies at most two unknowns can be solved per equation.

Graphical analysis approach Known vector quantities are drawn head to tail using a convenient scale. Unknown vectors, which complete the polygon, will be drawn last. Their magnitudes and directions will be measured directly from the drawing. This method is appropriate when expressing the vector quantities algebraically results in an awkward mathematical expressions.

Vector/Graphic approach Here, both the vector algebra and graphical techniques are used altogether to determine the solutions in the most convenient way. Sketch of the vector polygon representing the vector equation is helpful. With the sketch, we may see the convenient directions along which the projection of the vectors yields the simple scalar component equation. Ultimately, simultaneous equations may be avoided by a careful choice of the projecting directions.

Example 9.13 ([3], SP. 5/7) The wheel of radius $r = 300$ mm rolls to the right without slipping and has a velocity $v_0 = 3$ m/s of its center O. Calculate the velocity of point A on the wheel for the instant represented.

Solution: Velocity of A can be computed from O had the angular velocity of the wheel been known. Since the wheel rolls to the right without slipping, the velocity at C is zero. This can be used to determine the angular velocity.

$[\omega = v_O/r]$ $\qquad\qquad$ $\omega = 3/0.3 = 10 \ \text{rad/s} \ \text{CW}$

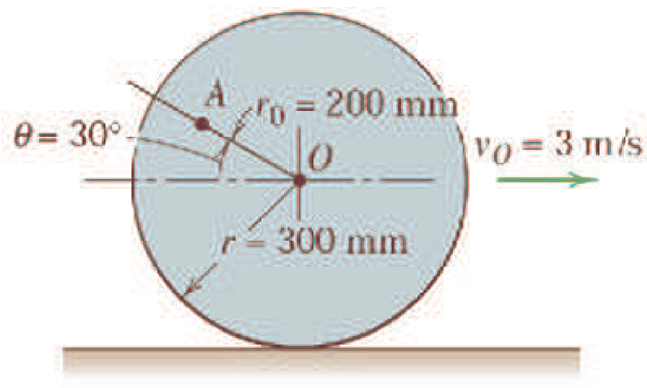


Figure 9.26: Example 9.13 ([3], pp. 359)

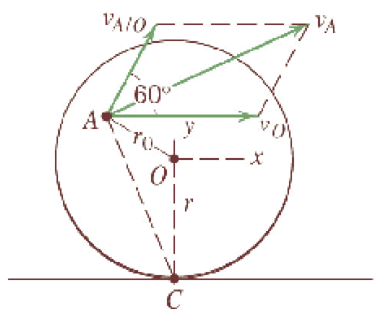


Figure 9.27: Solution to example 9.13

Another way to compute $\mathbf{v}_A$ is to compute from $\mathbf{v}_C$ which is $\mathbf{O}$. That is

$$[\mathbf{v}_A = \mathbf{v}_C + \omega \times \mathbf{r}_{A/C}] \qquad \mathbf{v}_A = \mathbf{0} + 10\mathbf{k} \times (-0.2\cos 30\mathbf{i} + (0.3 + 0.2\sin 30)\,\mathbf{j})$$

$$v_A = 4.36 \quad \text{m/s} \quad \angle 23.4^\circ$$

Example 9.14 ([3], SP. 5/10) The power screw turns at a speed that gives the threaded collar C a velocity of 0.25 m/s vertically down. Determine the angular velocity of the slotted arm when $\theta = 30^\circ$.

Solution: Imagine there are two points: point A on the slotted arm and point B on the collar. A and B coincide at $\theta = 30^\circ$. Because of the sliding contact constraint from the slot, $\mathbf{v}_{A/B}$ has the direction along the slot (away from O). Also $\mathbf{v}_A$ is always perpendicular to the link $\overline{AO}$. See fig. 9.29.

$$[\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}] \qquad v_A = 0.25\cos 30 = 0.217 \quad \text{m/s}$$

$$[\omega = v/r] \qquad \omega = v_A/\overline{OA} = 0.417 \quad \text{rad/s} \quad \text{CCW}$$

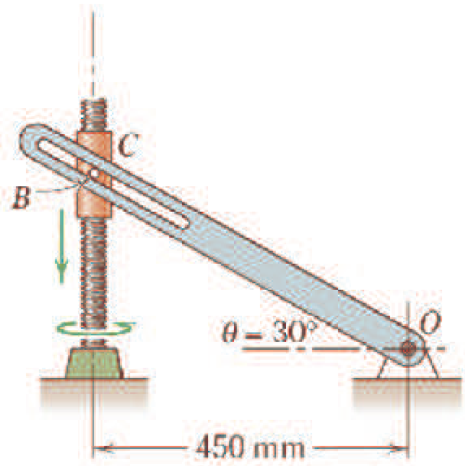


Figure 9.28: Example 9.14 ([3], pp. 362)

Figure 9.29: Solution to example 9.14

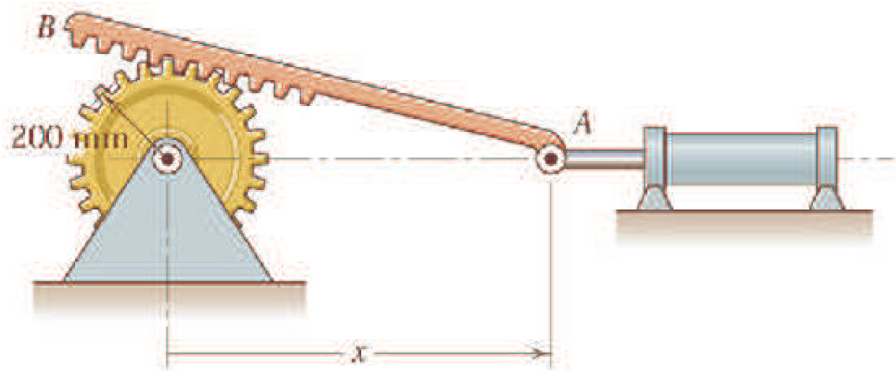


Figure 9.30: Example 9.15 ([3], pp. 367)

Example 9.15 ([3], Prob. 5/78) The rotation of the gear is controlled by the horizontal motion of end A of the rack AB. If the piston rod has a constant velocity 300 mm/s during a short interval of motion, determine the angular velocity of the gear and the angular velocity of AB at the instant when $x = 800$ mm.

Solution: Velocity at A propagates over the same body to C, which is the point on the rack AB at where it makes a contact with the pitch of the gear. Here point C is constrained to have the same velocity as the coincident point on the gear (but not the acceleration). Or the gear teeth will be broken. Velocity then passes from one body to the attached body. In particular, the gear rotates.

Since the velocity of a point on the rotating gear is always perpendicular to the radial line, it implies that

$$\mathbf{v}_C \perp \overline{OC}$$

From the geometry of the triangle OCA (fig. 9.31),

$$\theta = \sin^{-1}(0.2/0.8) = 14.48^\circ$$

Setting up the velocity equation and draw the relevant velocity polygon (fig. 9.31), we have

$$[\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{C/A}] \qquad v_C = 0.3\cos\theta = \omega_O \times 0.2, \quad \omega_O = 1.45 \text{ rad/s CW}$$

$$v_{C/A} = 0.3\sin\theta = \omega_{AB} \times \sqrt{0.8^2 - 0.2^2}, \quad \omega_{AB} = 0.0968 \text{ rad/s CCW}$$

Example 9.16 ([3], Prob. 5/83) The flywheel turns CW with a constant speed of 600 rev/min, and the connecting rod AB slides through the pivoted collar at C. For the position of $\theta = 45^\circ$, determine the angular velocity of AB using the relative velocity relations. Choose a point D on AB coincident with C as a reference point whose direction of velocity is known.

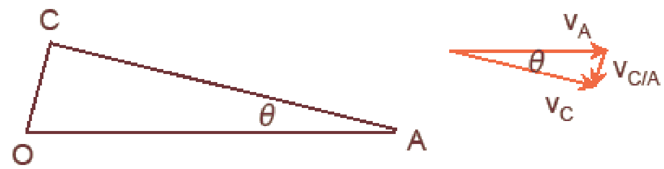


Figure 9.31: Solution to example 9.15

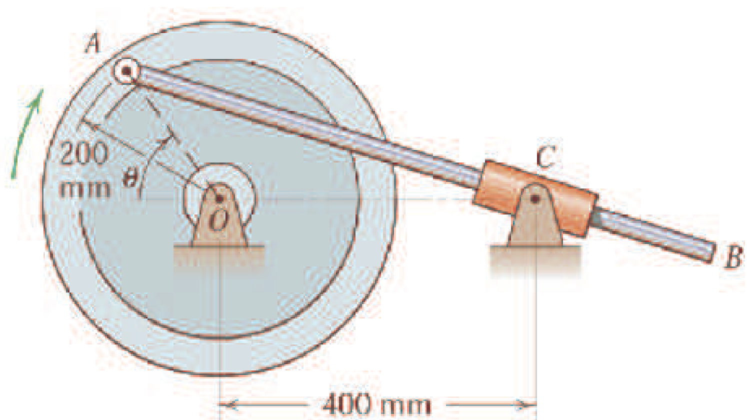


Figure 9.32: Example 9.16 ([3], pp. 368)

Solution: Choose D to be the point on the linkage AB which coincides with the pivot point C on the collar. Using the relative velocity equation,

$$[\mathbf{v}_D = \mathbf{v}_C + \mathbf{v}_{D/C}] \qquad \mathbf{v}_D = \mathbf{v}_{D/C} \text{ along the slot and pointing out of } A$$

$\mathbf{v}_A$ can be determined from either the motion of the flywheel or the motion of link AB. From this relationship and the velocity polygon in fig. 9.33, ω_{AB} can be found.

$$[\mathbf{v}_A = \mathbf{v}_D + \mathbf{v}_{A/D}] \qquad v_A = r\omega = 0.2 \times (600 \times 2\pi/60) = 12.566 \text{ m/s}$$

$$v_{A/D} = v_A \sin 59.63 = 10.84 = \omega_{AB} \times 0.56$$

$$\omega_{AB} = 19.36 \text{ rad/s CW}$$

Example 9.17 ([4], Prob. 5/85) The Geneva mechanism is shown again here. By relative motion principle, determine the angular velocity of wheel C for $\theta = 20°$. Wheel A has a constant CW angular velocity $\omega_1 = 2$ rad/s.

Solution: Let P be the point on the wheel A at the knob, while Q be the

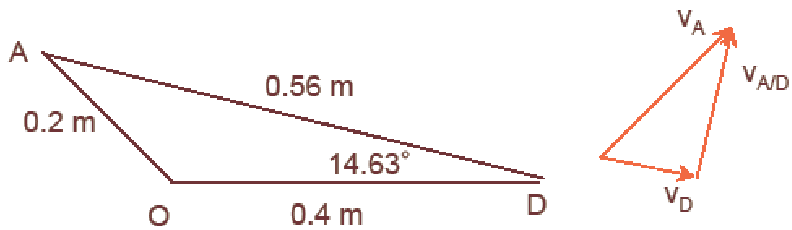


Figure 9.33: Solution to example 9.16

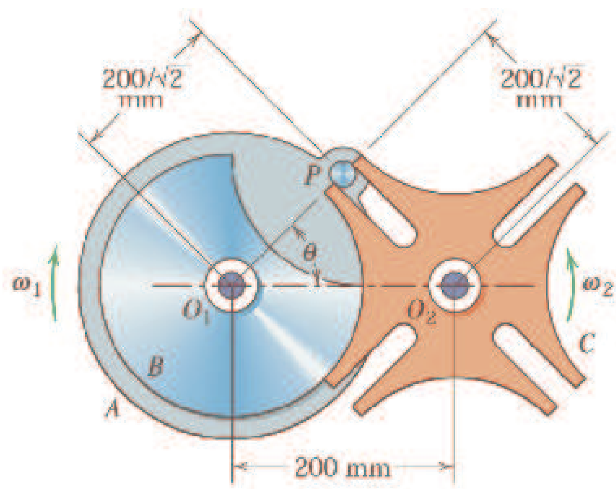


Figure 9.34: Example 9.17 ([4], pp. 360)

point on the wheel C that is coincident with P when $\theta = 20°$. Their relative velocity is constrained to be along the slot on the wheel C. Therefore

$$[\mathbf{v}_P = \mathbf{v}_Q + \mathbf{v}_{P/Q}] \qquad v_P = \omega \times r = 2 \times \left(0.2/\sqrt{2}\right) = 0.283 \text{ m/s}$$

$$v_Q = v_P \sin 34.355 = \omega_2 \times 0.083, \quad \omega_2 = 1.924 \text{ rad/s CCW}$$

Example 9.18 ([3], Prob. 5/91) At the instant represented, $a = 150$ mm and $b = 125$ mm, and the distance $a + b$ between A and C is decreasing at the rate of 0.2 m/s. Determine the common velocity v of points B and D for this instant.

Solution: Because B and D are on the same rigid body translating along the slot, $\mathbf{v}_B = \mathbf{v}_D$. The velocity relationship is obvious:

$$[\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}] \qquad [\mathbf{v}_D = \mathbf{v}_C + \mathbf{v}_{D/C}]$$

From the statement, block A and C become closer which means $\dot{a} + \dot{c} = -0.2$. And from the imposing mechanism, A moves to the right and B to the left.

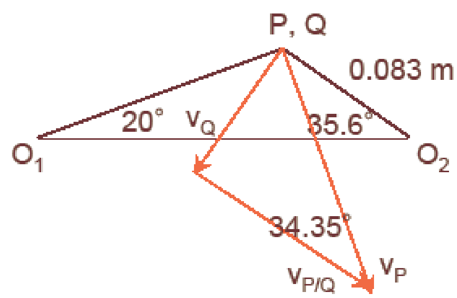


Figure 9.35: Solution to example 9.17

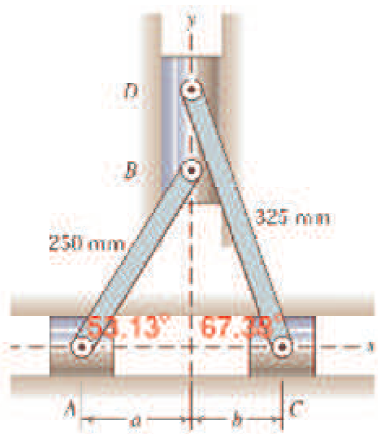


Figure 9.36: Example 9.18 ([3], pp. 370)

Therefore

$$\mathbf{v}_A - \mathbf{v}_C = 0.2\mathbf{i} \ \ \text{m/s}$$

Referring to the velocity diagram in fig. 9.37,

$$\frac{v_B}{\tan 22.62} + \frac{v_B}{\tan 36.87} = 0.2, \quad \mathbf{v}_B = 0.0536\mathbf{j} \ \ \text{m/s}$$

Example 9.19 ([3], Prob. 5/92) The wheel rolls without slipping. For the instant portrayed, when O is directly under point C, link OA has a velocity $v = 1.5$ m/s to the right and $\theta = 30°$. Determine the angular velocity ω of the slotted link.

Solution: Since the wheels rolls without slipping,

$[v_O = \omega r]$ $$1.5 = \omega_O \times 0.1, \quad \omega_O = 15 \ \ \text{rad/s CW}$$

$$v_P = \omega_O \times (2 \times 0.1 \cos 15) = 2.9 \ \ \text{m/s}$$

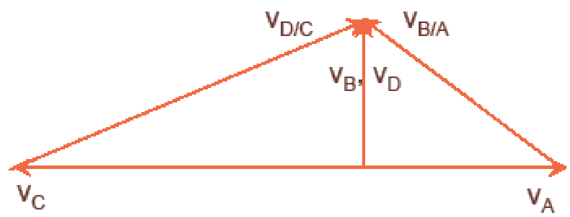


Figure 9.37: Solution to example 9.18

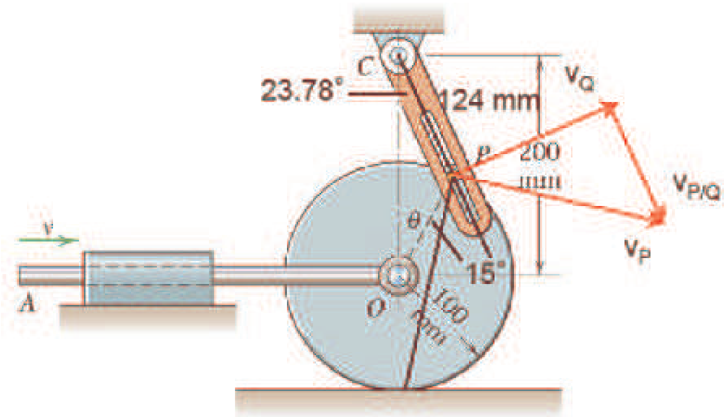


Figure 9.38: Example and solution of 9.19 ([3], pp. 370)

Let P be the point at the pin on the disk and Q be the coincident point on the slotted arm. According to the figure, their relative velocity directs along the slot:

$$\left[\mathbf{v}_P = \mathbf{v}_Q + \mathbf{v}_{P/Q}\right] \qquad v_Q = v_P \cos(15 + 23.78) = 2.26 \ \text{ m/s}$$

$$[\omega = v/r] \qquad \omega_C = 2.26/0.124 = 18.23 \ \text{ rad/s} \ \ \text{CCW}$$

Example 9.20 ([4], Prob. 5/88) Ends A and C of the connected links are controlled by the vertical motion of the piston rods of the hydraulic cylinders. For a short interval of motion, A has an upward velocity of 3 m/s, and C has a downward velocity of 2 m/s. Determine the velocity of B for the instant when $y = 150$ mm.

Solution: $\mathbf{v}_B$ can be calculated from either $\mathbf{v}_C$ or $\mathbf{v}_A$ according to

$$\mathbf{v}_B = \mathbf{v}_C + \mathbf{v}_{B/C} = \mathbf{v}_A + \mathbf{v}_{B/A}$$

The related velocity diagram is shown in fig. 9.40, from which, by the law of

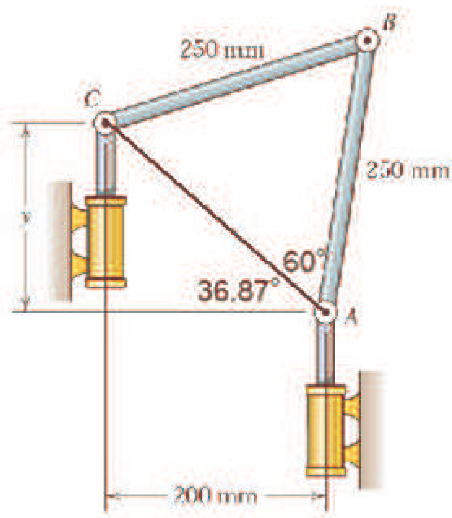


Figure 9.39: Example 9.20 ([4], pp. 361)

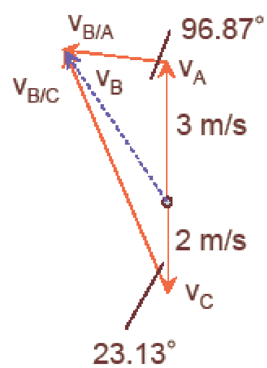


Figure 9.40: Solution to example 9.20

sine, the following relation can be written:

$$\frac{5}{\sin 60} = \frac{v_{B/C}}{\sin 96.87} = \frac{v_{B/A}}{\sin 23.13}$$

Therefore,

$$v_{B/C} = 5.732 \text{ m/s} \text{ and } v_{B/A} = 2.268 \text{ m/s}$$

Using the Pythagorus's theorem, we have

$$\left(v_{B/C} \sin 23.13\right)^2 + \left(3 + v_{B/A} \cos 83.13\right)^2 = v_B^2$$

$$v_B = 3.97 \text{ m/s} \text{ directed along the dotted arrow}$$

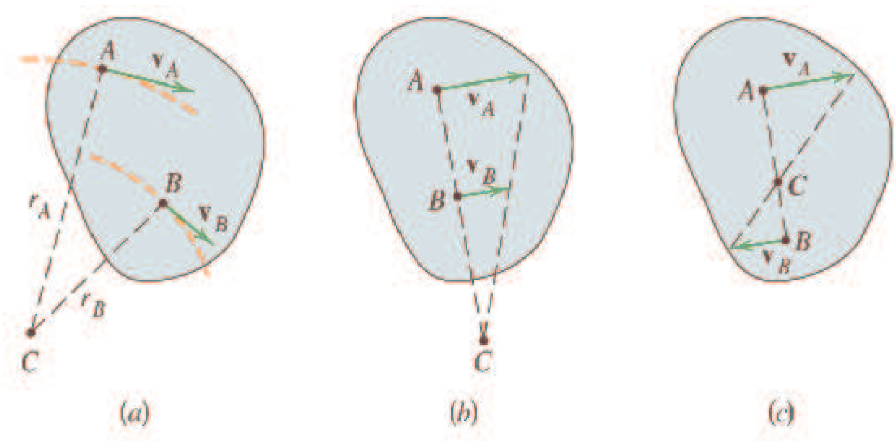


Figure 9.41: Determination of the ICZV point ([3], pp. 371)

9.5 Instantaneous Center of Zero Velocity

Principle of relative motion determines the velocity of a point on a rigid body by adding the relative velocity, due to the rotation about a reference point, to the velocity of the reference point. *If the reference point has zero velocity momentarily, the body may be considered to be in pure rotation about an axis, normal to the plane of motion, passing through this point.* This point is called the instantaneous center of zero velocity or ICZV in short. It greatly helps in visualizing and analyzing velocity in the plane motion. Note that ICZV has zero velocity but *the acceleration may not be zero.*

Figure 9.41 illustrates the determination of the ICZV point. In (a), point A and B, instantaneously, have the *absolute* circular motion about point C. Therefore point C is the ICZV. However, it may not be on the body physically. We still can visualize it as the point lying on the extended body. It is important to denote that ICZV is not a fixed point in the body nor a fixed point in the plane. After locating the ICZV, the body angular velocity is calculated as

$$\omega = \frac{v_A}{r_A} = \frac{v_B}{r_B} \tag{9.15}$$

Or the velocity relationship between any two points is

$$v_B = \left(\frac{r_B}{r_A}\right) v_A \tag{9.16}$$

In case the body motion is just the translation motion, the ICZV is located at infinity along the perpendicular line to the velocity.

Example 9.21 ([4], Prob. 5/115) Vertical oscillation of the spring-loaded plunger F is controlled by a periodic change in pressure in the vertical hydraulic

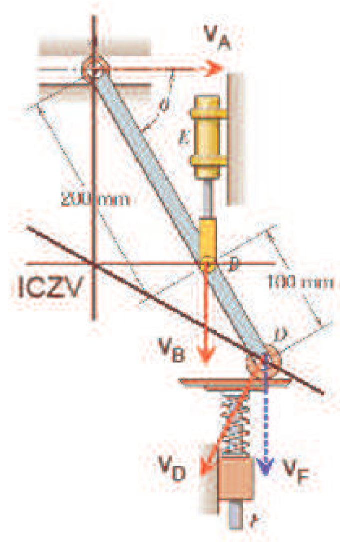


Figure 9.42: Example and solution of 9.21 ([4], pp. 368)

cylinder E. For the position $\theta = 60°$, determine the angular velocity of AD and the velocity of the roller A in its horizontal guide if the plunger F has a downward velocity of 2 m/s.

Solution: In this configuration, $\mathbf{v}_B$ is forced to move downward by the hydraulic while $\mathbf{v}_A$ is forced to move horizontally by the guide. Therefore the ICZV of the linkage ABD is at point C. Its consequence is the direction of $\mathbf{v}_D$ is known to be perpendicular to $\overline{CD}$.

Because the plunger moves downward with the rate of 2 m/s, the *vertical component* of $\mathbf{v}_D$ is 2 m/s downward. Therefore

$$v_D = \frac{2}{\cos 30} = \omega_{AD} \times (2 \times 0.1 \cos 30)$$

$$\omega_{AD} = 13.33 \quad \text{rad/s} \quad \text{CW}$$

Using the ICZV,

$$v_A = \omega_{AD} \times (0.2 \sin 60) = 2.309 \quad \text{m/s, to the right}$$

Example 9.22 ([3], Prob. 5/121) Determine the angular velocity ω of the ram head AE of the rock crusher in the position for which $\theta = 60°$. The crank OB has an angular speed of 60 rev/min. When B is at the bottom of its circle, D and E are on a horizontal line through F, and lines BD and AE are vertical. The dimensions are $OB = 100$ mm, $BD = 750$ mm, and $AE = ED = DF = 375$ mm. Carefully construct the configuration graphically, and use the method of

ICZV.

Solution: Graphical method using ICZV is the best way for this problem, which is capture in fig. 9.43. Here we explain another approach using the vector/graphic method. From the initial and current posture, write the distance between point O and F:

$$\begin{aligned} 100\sin 60 - 750\cos\alpha + 375\cos\beta &= 375 \\ -100\cos 60 + 750\sin\alpha + 375\sin\beta &= 850 \end{aligned}$$

By the help of complex exponent, $\alpha = 85.85°$, $\beta = 23.9°$.

Now, write the distance between point O and A:

$$\begin{aligned} 375\cos\theta + 375\cos\zeta + 750\cos\alpha - 100\sin 60 = 375 \\ -375\sin\theta + 375\sin\zeta + 750\sin\alpha - 100\cos 60 = 475 \end{aligned}$$

By the help of complex exponent, $\theta = 81.04°$, $\zeta = 21.49°$.

Next, find the distance from the ICZV to point of interest. See fig. 9.43 and fig. 9.44.

$$\begin{aligned} l_1\sin 30 &= -100\cos 60 + 750\sin\alpha - l_2\sin\beta \\ l_1\cos 30 &= l_2\cos\beta + 750\cos\alpha - 100\sin 60 \end{aligned}$$

$$l_1 = 773.63 \text{ mm}, \quad l_2 = 768.2 \text{ mm}$$

$$\begin{aligned} l_3\cos\beta &= l_4\cos\theta + 375\cos\zeta \\ l_4\sin\beta &= l_4\sin\theta - 375\sin\zeta \end{aligned}$$

$$l_3 = 435.8 \text{ mm}, \quad l_4 = 317.8 \text{ mm}$$

After locating the ICZV, the velocity of any point can be determined readily.

$$v_B = 60 \times 100 = (l_1 + 100) \times \omega_{BD}, \quad \omega_{BD} = 6.868 \text{ rev/min CW}$$

$$v_D = \omega_{BD} \times l_2 = 5276 \text{ mm} \cdot \text{rev/min} = l_3 \times \omega_{DE}, \quad \omega_{DE} = 12.1 \text{ rev/min CW}$$

$$v_E = \omega_{DE} \times l_4 = \omega_{AE} \times 375, \quad \omega_{AE} = 10.26 \text{ rev/min} = 1.07 \text{ rad/s CW}$$

Example 9.23 ([3], Prob. 5/122) The shaft at O drives the arm OA at a clockwise speed of 90 rev/min about the fixed bearing at O. Use the method of ICZV to determine the rotational speed of gear B (gear teeth not shown) if

(a) ring gear D is fixed and

(b) ring gear D rotates CCW about O with a speed of 80 rev/min.

Solution: It is easily observed that the motion of gear A is influenced by

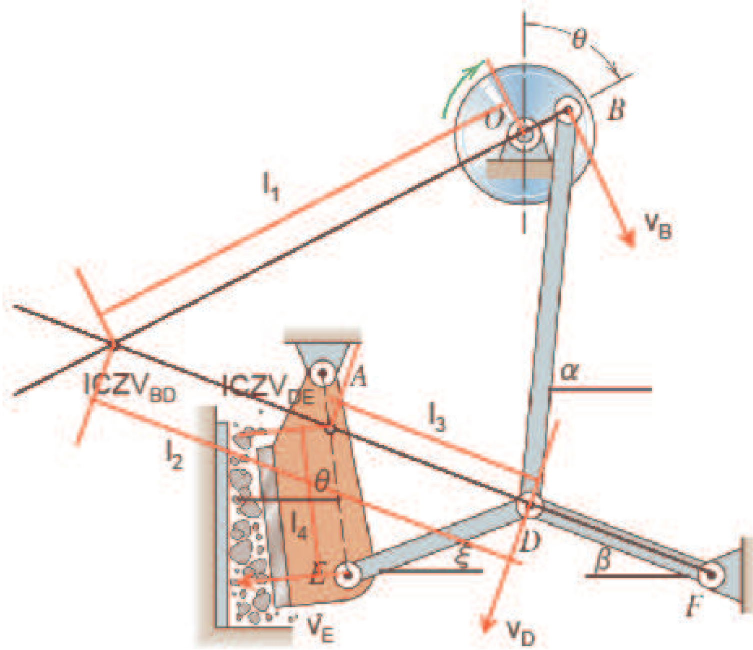


Figure 9.43: Example 9.22 ([3], pp. 380)

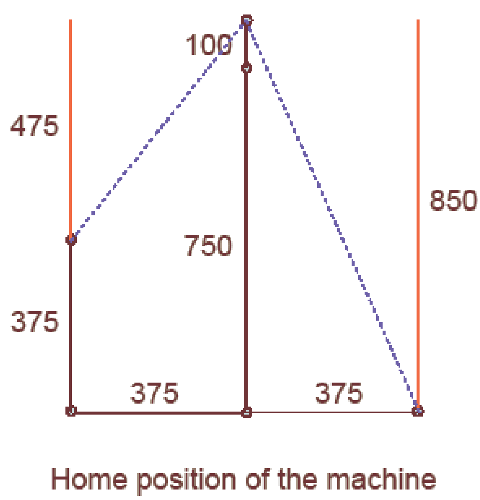


Figure 9.44: Solution to example 9.22

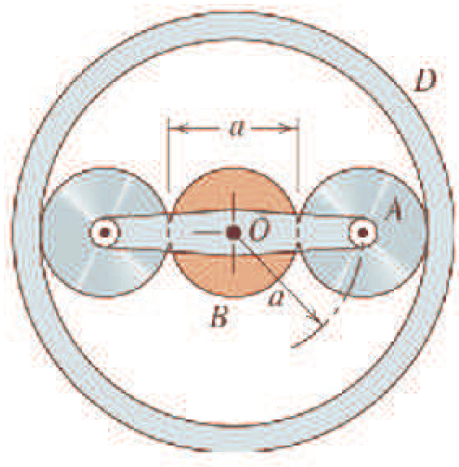


Figure 9.45: Example 9.23 ([3], pp. 380)

1. motion of link OA at point A: $v_{A_{\text{gear}}} = v_{A_{\text{link}}}$
2. motion of the leftmost point on the rim: $v_{C_{\text{gearA}}} = v_{C_{\text{gearB}}}$
3. motion of the rightmost point on the rim: $v_{D_{\text{gearA}}} = v_{D_{\text{gearD}}}$

When the ring gear D is fixed (see fig. 9.46),

$$v_A = 90\,a \downarrow$$

$$v_C = 2v_A = 2 \times 90\,a = \omega_B \times \frac{a}{2}, \quad \omega_B = 360 \ \text{ rev/min CW}$$

When the ring gear D rotates at 80 rev/min CCW (see fig. 9.46),

$$v_A = 90\,a \downarrow, \quad v_D = 80 \times \left(\frac{3a}{2}\right) = 120\,a \uparrow$$

$$v_C = \frac{10/7}{3/7}\,(90\,a) = \omega_B \times \frac{a}{2}, \quad \omega_B = 600 \ \text{ rev/min CW}$$

Example 9.24 ([4], Prob. 5/118) The large roller bearing rolls to the left on its outer race with a velocity of its center O of 0.9 m/s. At the same time, the central shaft and inner race rotate CCW with an angular speed of 240 rev/min. Determine the angular velocity of each of the rollers.

Solution: The vertical velocity profile is imposed on fig. 9.47. From the problem, it is specified that $v_O = 0.9$ m/s and $\omega_i = 240$ rev/min $= 8\pi$ rad/s CCW.

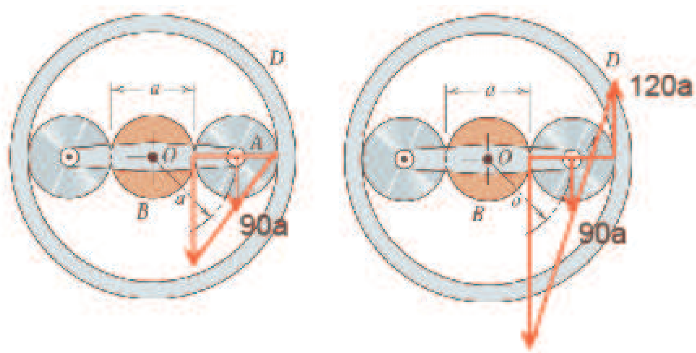


Figure 9.46: Solution to example 9.23

From the velocity profile, the ICZV of the inner race is $\frac{0.9}{8\pi}$ m lower to point O. Therefore, velocity of the point on the roller contacting with the inner race is

$$v = 8\pi \times \left(0.05 - \frac{0.9}{8\pi}\right) = 0.3566 \quad \text{m/s} \rightarrow$$

If we imagine the bearing as one rigid disk rolling with the same angular velocity. That value is

$$\omega_O = \frac{0.9}{0.125} = 7.2 \quad \text{rad/s CCW}$$

which is also the angular velocity of the outer ring. Hence, velocity of the point on the roller contacting with the outer race is

$$v = 7.2 \times 0.025 = 0.18 \quad \text{m/s} \leftarrow$$

Consider the velocity profile across the roller. Using the similar triangle relationship,

$$\frac{0.18}{0.3566} = \frac{x}{0.05 - x}, \quad x = 16.77 \quad \text{mm}$$

Therefore, its angular velocity is

$$\omega_{\text{roller}} = \frac{0.18}{x} = 10.732 \quad \text{rad/s CW}$$

9.6 Relative Acceleration

The relative acceleration relationship with non-rotating reference axes can be obtained from differentiating the relative velocity relation with respect to time, i.e.

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} \tag{9.17}$$

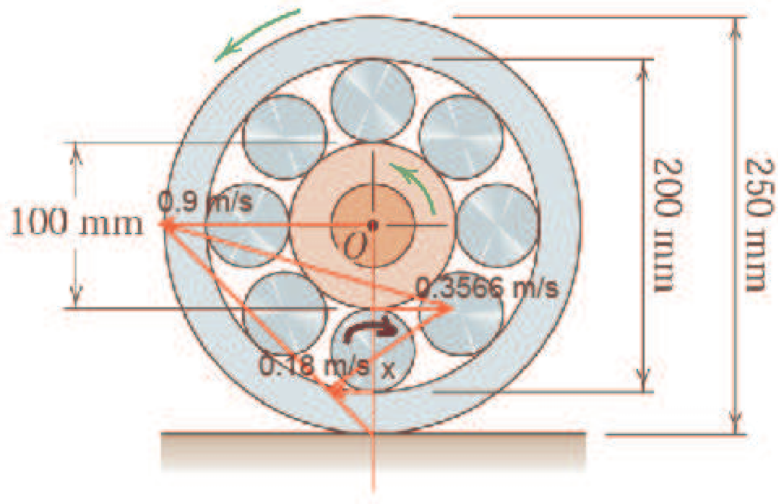


Figure 9.47: Example and solution of 9.24 ([4], pp. 369)

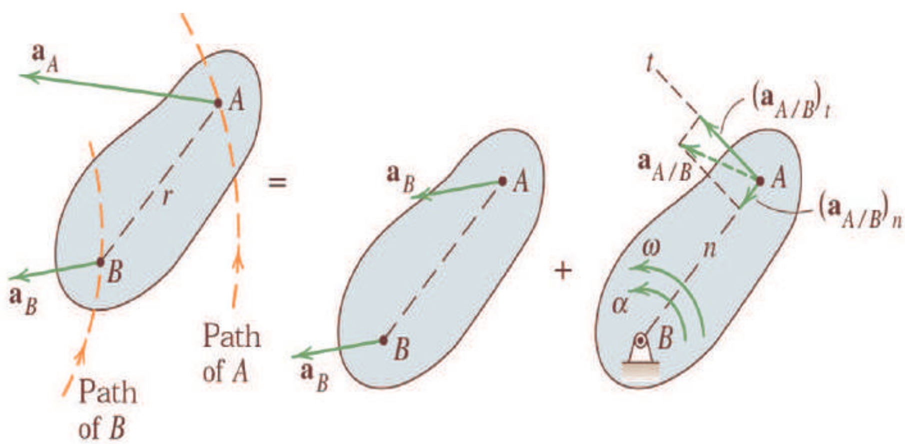


Figure 9.48: Relative acceleration between point A and B ([3], pp. 381)

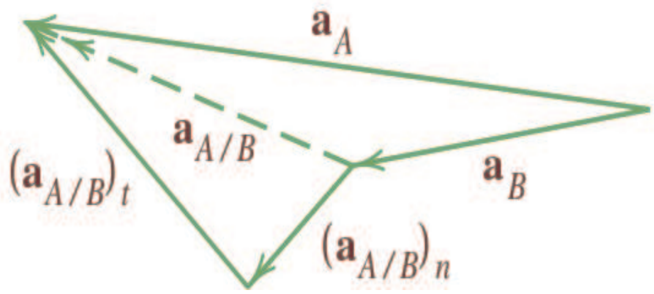


Figure 9.49: Acceleration diagram ([3], pp. 381)

where $\mathbf{a}_{A/B}$ is the acceleration that A appears to have to a non-rotating observer moving with B.

If A and B are two points on the same rigid body in plane motion, the distance between them remains constant so that *the observer moving with B perceives A to have circular motion about B*. As a result, the relative acceleration term can be partitioned into the normal and the tangential components. Normal component acceleration, which is directed from A toward B, is due to the change of direction of $\mathbf{v}_{A/B}$. Tangential component acceleration, which is perpendicular to $\overline{AB}$, is due to the change in magnitude of $\mathbf{v}_{A/B}$. See fig. 9.48 and fig. 9.49 for the illustration. Similar derivation for the velocity equation can be applied to determine the acceleration equation, for which the result would be

$$\mathbf{a}_A = \mathbf{a}_B + \left(\mathbf{a}_{A/B}\right)_n + \left(\mathbf{a}_{A/B}\right)_t \tag{9.18}$$

where

$$\left(\mathbf{a}_{A/B}\right)_n = \omega \times (\omega \times \mathbf{r}), \qquad \left(a_{A/B}\right)_n = r\omega^2 = v_{A/B}^2/r \tag{9.19}$$

$$\left(\mathbf{a}_{A/B}\right)_t = \alpha \times \mathbf{r}, \qquad \left(a_{A/B}\right)_t = r\alpha = \dot{v}_{A/B} \tag{9.20}$$

Methods in solving the relative acceleration equation

There are at least three ways in solving the relative acceleration equation. Each of which has its pros and cons as briefly described below.

1. Vector algebra approach
2. Graphical analysis approach
3. Vector/Graphic approach

Vector algebra approach Each term in the equation is represented in the selected coordinate system, which may be $x-y$, $n-t$, or $r-\theta$ coordinates. For two dimensional problems, each vector equation has two independent scalar equations. Therefore at most two unknowns can be solved per equation. This method is appropriate for writing the computer codes to solve the problems or when the exact analytical expression is needed.

Graphical analysis approach Known vectors are constructed using a convenient scale. They are connected head to tail according to the vector equation. Finally, unknown vectors which complete the polygon are then measured directly from the drawing. This approach is suitable when the vector terms result in an awkward mathematical expression.

Vector/Graphic approach This approach combines the good points of the vector and the graphical methods together. The scalar component equations may be written by projecting the vectors along the convenient directions. After that a set of the equations may be solved simultaneously. However, this may be avoided by a careful choice of the projecting direction.

It is recommended for every approach of solving two dimensional problems to sketch the *vector polygon* representing the vector equation. Known and unknown quantities with the constraints can be visualized explicitly. One final note before ending this section. Because $\mathbf{a}_n$ depends on the velocity, usually it is required to solve for the velocity before the acceleration calculation can be made.

Example 9.25 ([4], Prob. 5/138) If the wheel in each case rolls on the circular surface without slipping, determine the acceleration of the point C on the wheel momentarily in contact with the circular surface. The wheel has an angular velocity ω and an angular acceleration α.

Solution: Because the wheel rolls without slipping, $\mathbf{v}_C = 0$. For both cases, point O moves along the circular path. Furthermore, the rotational motion of the wheel, ω and α, is given. Therefore,

$$\mathbf{v}_O = r\omega\mathbf{i}, \qquad (\mathbf{a}_O)_t = r\alpha\mathbf{i}$$

Applying these expressions to the acceleration equation for each case, we have

(a) $\left[\mathbf{a}_C = \mathbf{a}_O + \mathbf{a}_{C/O}\right]$

$$\begin{aligned}\mathbf{a}_C &= (\mathbf{a}_O)_t + (\mathbf{a}_O)_n + \left(\mathbf{a}_{C/O}\right)_t + \left(\mathbf{a}_{C/O}\right)_n \\ &= r\alpha\mathbf{i} + \frac{r^2\omega^2}{R-r}\mathbf{j} - r\alpha\mathbf{i} + r\omega^2\mathbf{j} = r\omega^2\left(\frac{R}{R-r}\right)\mathbf{j}\end{aligned}$$

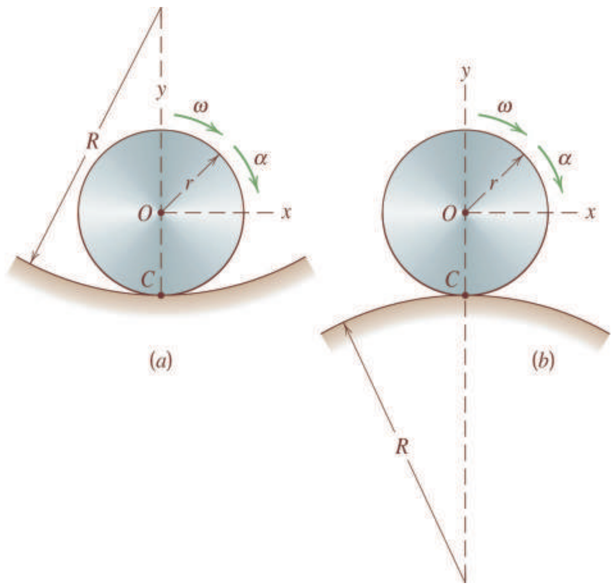


Figure 9.50: Example and solution of 9.25 ([4], pp. 379)

$$|\mathbf{a}_C| = r\omega^2 \left(\frac{R}{R-r}\right) > r\omega^2$$

(b) $\left[\mathbf{a}_C = \mathbf{a}_O + \mathbf{a}_{C/O}\right]$

$$\begin{aligned}\mathbf{a}_C &= (\mathbf{a}_O)_t + (\mathbf{a}_O)_n + \left(\mathbf{a}_{C/O}\right)_t + \left(\mathbf{a}_{C/O}\right)_n \\ &= r\alpha\mathbf{i} - \frac{r^2\omega^2}{R+r}\mathbf{j} - r\alpha\mathbf{i} + r\omega^2\mathbf{j} = r\omega^2\left(\frac{R}{R+r}\right)\mathbf{j}\end{aligned}$$

$$|\mathbf{a}_C| = r\omega^2 \left(\frac{R}{R+r}\right) < r\omega^2$$

Example 9.26 ([4], Prob. 5/144) The simplified clam-shell bucket is shown. With the block at O considered fixed and with the constant velocity of the control cable at C equal to 0.5 m/s, determine the angular acceleration α of the right-hand bucket jaw when $\theta = 45°$ as the bucket jaws are closing.

Solution: The rotational motion of the bucket jaw is to be determined from the relative motion between two points B and C on the jaw.

Velocity analysis Here it may be simpler to apply the ICZV method. First, locate the ICZV of the right jaw. Since the velocity direction of B and C are known, ICZV is the intersection of the radial lines as shown in fig. 9.52. Let that ICZV point be called point D. Then, for the given configuration $\theta = 45°$,

$$\overline{CD} = \overline{CO}\tan 50.345 = 692.78 \text{ mm}$$

On the bucket,

$$[\omega = v/r] \qquad \omega_{BC} = 0.5/\overline{CD} = 0.7212 \text{ rad/s CW closing}$$

$$[v = \omega r] \qquad v_B = \omega_{BC} \times \left(\overline{CO}/\cos 50.345 - \overline{BO}\right) = 0.2165 \text{ m/s}$$

On the linkage $\overline{BO}$,

$$[\omega = v/r] \qquad \omega_{BO} = v_B/\overline{BO} = 0.361 \text{ rad/s CCW}$$

Acceleration analysis The relative acceleration between point B and C are constrained by the bucket to be the circular motion. Since point C moves upward with constant velocity, $\mathbf{a}_C = \mathbf{0}$. Making use of the n-t description, we have

$$\left[\mathbf{a}_B = \mathbf{a}_C + \mathbf{a}_{B/C}\right] \qquad (\mathbf{a}_B)_n + (\mathbf{a}_B)_t = \left(\mathbf{a}_{B/C}\right)_n + \left(\mathbf{a}_{B/C}\right)_t$$

where

$$\left(a_{B/C}\right)_n = \overline{BC}\omega_{BC}^2 = 0.2604 \text{ m/s}^2 \qquad (a_B)_n = \overline{BO}\omega_{BO}^2 = 0.0782 \text{ m/s}^2$$

Substitute these values in the above relationship. From the acceleration diagram in fig. 9.52 along the horizontal direction, we have

$$0.2604\cos 22.5 + \left(a_{B/C}\right)_t \sin 22.5 - (a_B)_t \cos 50.345 - 0.0782\sin 50.345 = 0$$

Along the vertical direction, we have

$$0.2604\sin 22.5 - \left(a_{B/C}\right)_t \cos 22.5 - (a_B)_t \sin 50.345 + 0.0782\cos 50.345 = 0$$

Hence

$$\left(a_{B/C}\right)_t = -0.049 \text{ m/s}^2 \text{ (wrong direction)} \qquad (a_B)_t = 0.2532 \text{ m/s}^2$$

Therefore

$$[\alpha = a_t/r] \qquad \alpha_{BC} = 0.049/0.5 = 0.098 \text{ rad/s}^2 \text{ CW}$$

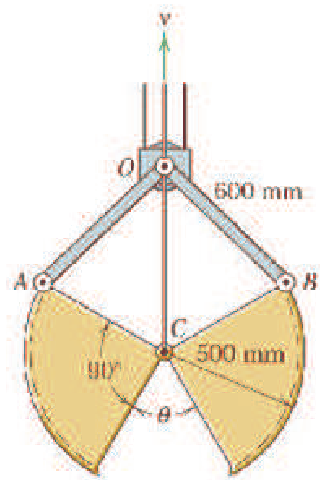


Figure 9.51: Example 9.26 ([4], pp. 381)

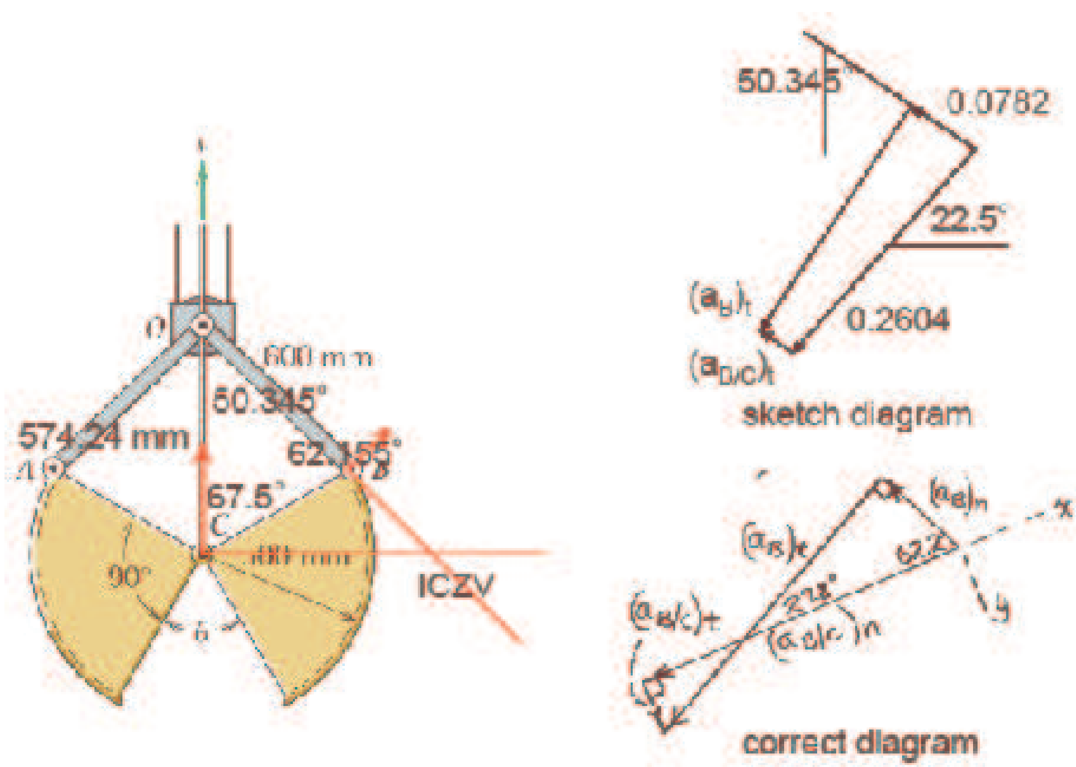


Figure 9.52: Solution to example 9.26

Example 9.27 ([3], Prob. 5/152) The mechanism where the flexible band F attached to the sector at E is given a constant velocity of 4 m/s as shown. For the instant when BD is perpendicular to OA, determine the angular acceleration of BD.

Solution: Motion of the flexible band causes the rotation of the sector, which then causes the motion of the link AD and BD, respectively. The sector rotates about O by the pulling force of the band. Consequently, point E on the sector moves in circular path about O with the velocity of 4 m/s; the velocity of the band. Hence, at this instant,

$[\omega = v/r]$ $$\omega_{OA} = 4/0.2 = 20 \ \text{rad/s CCW}$$

$[v = \omega r]$ $$v_A = \omega_{OA} \times 0.125 = 2.5 \ \text{m/s} \uparrow$$

Velocity analysis If we know the motion of A, the motion of D can be determined from the relative velocity and the useful velocity diagram shown in fig. 9.54 as

$[\mathbf{v}_D = \mathbf{v}_A + \mathbf{v}_{D/A}]$ $$v_D/v_A = 3/4, \quad v_D = 1.875 \ \text{m/s}$$

and

$$v_{D/A}/v_A = 5/4, \quad v_{D/A} = 3.125 \ \text{m/s}$$

Hence,

$[\omega = v/r]$ $$\begin{aligned} \omega_{BD} &= v_D/0.25 = 7.5 \ \text{rad/s CCW} \\ \omega_{AD} &= v_{D/A}/0.25 = 12.5 \ \text{rad/s CCW} \end{aligned}$$

Acceleration analysis Since the flexible band pulls the sector with a constant velocity, $(\mathbf{a}_E)_t = 0$. Therefore the angular acceleration of the sector is zero. This implies the total acceleration of A, $\mathbf{a}_A$, is in the normal direction pointing towards O. Consequently, $\mathbf{a}_D$ can be determined as

$[\mathbf{a}_D = \mathbf{a}_A + \mathbf{a}_{D/A}]$ $[a_n = r\omega^2]$

and the acceleration diagram shown in fig. 9.54,

$$\begin{aligned} a_A &= 0.125 \times 20^2 = 50 \ \text{m/s}^2 \ \leftarrow \\ (a_{D/A})_n &= 0.25 \times 12.5^2 = 39.0625 \ \text{m/s}^2 \\ (a_D)_n &= 0.25 \times 7.5^2 = 14.0625 \ \text{m/s}^2 \ \uparrow \end{aligned}$$

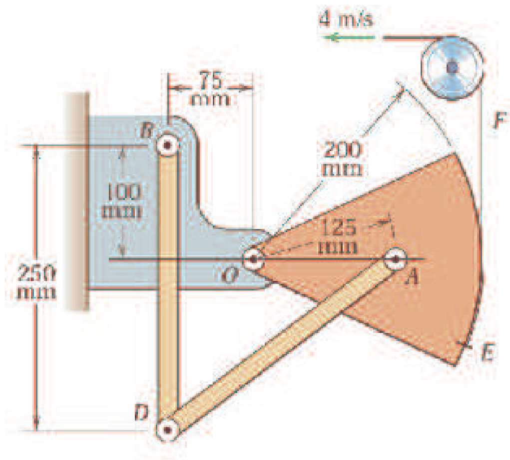


Figure 9.53: Example 9.27 ([3], pp. 392)

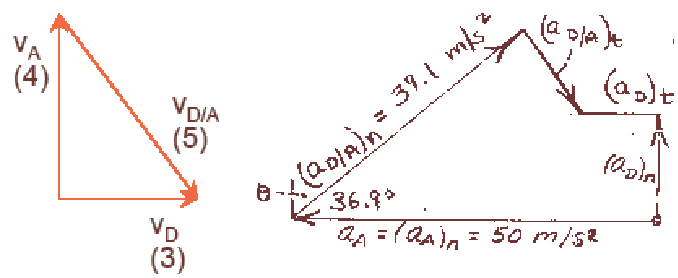


Figure 9.54: Solution to example 9.27

During constructing the acceleration diagram, it is acknowledged that

$$\left(\mathbf{a}_{D/A}\right)_t \perp \overline{AD} \quad \text{and} \quad (\mathbf{a}_D)_t \perp \overline{BD}$$

of which their magnitudes can be determined from the lengths of the polygon as

$$39.0625 \times \frac{3}{5} - \left(a_{D/A}\right)_t \times \frac{4}{5} - 14.0625 = 0, \qquad \left(a_{D/A}\right)_t = 11.72 \ \ \text{m/s}^2$$
$$39.0625 \times \frac{4}{5} + \left(a_{D/A}\right)_t \times \frac{3}{5} + (a_D)_t - 50 = 0, \qquad (a_D)_t = 11.72 \ \ \text{m/s}^2$$

Hence,

$$[\alpha = a_t/r] \qquad \begin{aligned} \alpha_{AD} &= \left(a_{D/A}\right)_t/0.25 = 46.875 \ \ \text{rad/s}^2 \ \ \text{CCW} \\ \alpha_{BD} &= (a_D)_t/0.25 = 46.875 \ \ \text{rad/s}^2 \ \ \text{CW} \end{aligned}$$

Example 9.28 ([3], Prob. 5/156) Elements of the switching device are shown. If the velocity v of the control rod is 0.9 m/s and is slowing down at the rate of

6 m/s^2 when $\theta = 60°$, determine the magnitude of the acceleration of C.

Solution: With the downward movement of the control rod, point A of the mechanism here is constrained to be always in contact with the horizontal surface. Therefore, direction of $\mathbf{v}_A$ and $\mathbf{a}_A$ are in horizontal. The other point B has obvious motion in vertical direction since it is connected with the control rod. See fig. 9.55.

Velocity analysis The configuration here suggests the use of ICZV for the analysis. Since the motion of point A and B on the linkage ABC are known, the ICZV point of ABC is the intersection of the perpendicular lines to their velocity as depicted in fig. 9.55.

$[\omega = v/r]$ $$\omega_{AB} = \frac{0.9}{0.075\cos 30} = 13.856 \text{ rad/s CCW}$$

Acceleration analysis The angular acceleration of link ABC can be determined from the acceleration relationship between point A and B as followed. See also the acceleration diagram in fig. 9.56.

$[\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}]$

With the vectors in the equation projected onto the vertical direction, the following equation can be written:

$$6 + 14.4\sin 30 - (a_{A/B})_t \sin 60 = 0$$

With the vectors in the equation projected onto the horizontal direction, the following equation can be written:

$$a_A = 14.4\cos 30 + (a_{A/B})_t \cos 60$$

Solving these two simultaneous equations,

$$(a_{A/B})_t = 15.242 \text{ m/s}^2, \quad a_A = 20.09 \text{ m/s}^2$$

$[\alpha = a_t/r]$ $$\alpha_{AB} = (a_{A/B})_t/\overline{AB} = 203.227 \text{ rad/s}^2 \text{ CW}$$

Then, $\mathbf{a}_C$ can be directly determined from the following relationship. See the acceleration diagram in fig. 9.56.

$[\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}]$ $$(a_{C/B})_t = 15.242 \text{ m/s}^2, \quad (a_{C/B})_n = 14.4 \text{ m/s}^2$$

$$\begin{aligned}\mathbf{a}_C &= (14.4\cos 30 + 15.242\cos 60)\,\mathbf{i} + (6 - 14.4\sin 30 + 15.242\sin 60)\,\mathbf{j} \\ &= 20.09\mathbf{i} + 12.0\mathbf{j}, \quad a_C = 23.4 \text{ m/s}^2\end{aligned}$$

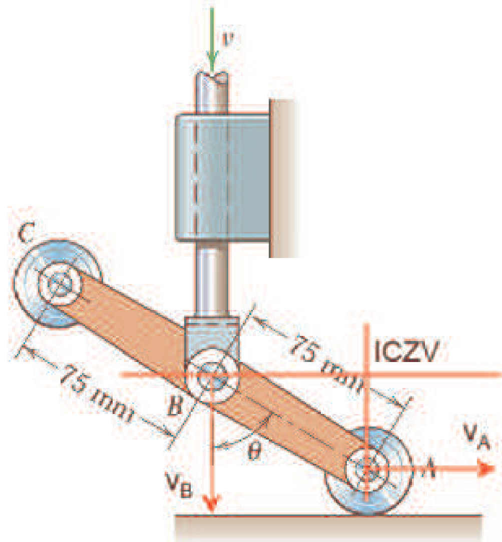


Figure 9.55: Example 9.28 ([3], pp. 393)

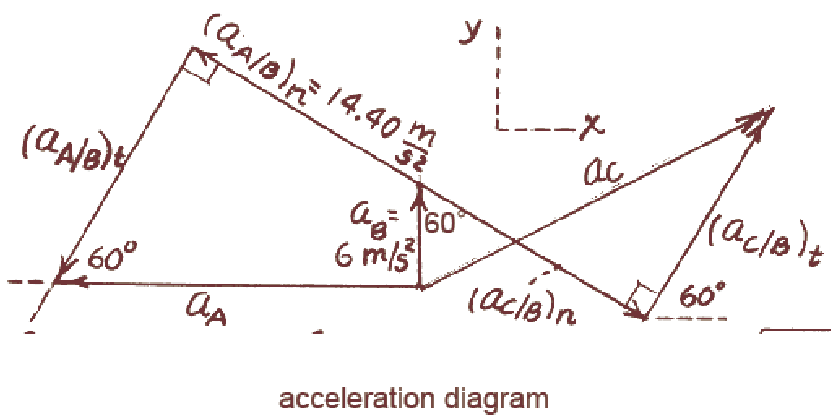


Figure 9.56: Solution to example 9.28

Example 9.29 ([4], Prob. 5/149) An oil pumping rig is shown in the figure. The flexible pump rod D is fastened to the sector at E and is always vertical as it enters the fitting below D. The link AB causes the beam BCE to oscillate as the weighted crank OA revolves. If OA has a constant CW speed of 1 rev every 3 s, determine the acceleration of the pump rod D when the beam and the crank OA are both in the horizontal position shown.

Solution: The flexible pump rod D and the sector E are connected such that the motion of the sector in the tangential direction is imparted to the rod entering and exiting vertically the hole tube. According to the motion transmitted by the linkage OA, AB, and BC forming the four bar linkage mechanism, the sector itself moves back and forth along the specific circular arc. To determine the acceleration of the pumping rod, kinematic analysis along the motion transmission from the crank OA should be perform starting from the velocity-level analysis as follow.

Velocity analysis First, let us convert the unit of the angular velocity of the crank OA.

$$\omega_{OA} = \frac{1}{3} \text{ rev/s} = \frac{2\pi}{3} \text{ rad/s CW constant}$$

At the position shown in fig. 9.57, the beam BCE and the crank OA are both in the horizontal position, for which the velocity at A can be simply calculated as

$$\mathbf{v}_A = \omega_{OA} \times \overline{OA} = 0.4\pi \text{ m/s } \uparrow$$

In this case, ω_{AB} and ω_{CE} can be conveniently determined using the method of ICZV. Since the direction of $\mathbf{v}_A$ and $\mathbf{v}_B$ are known, the location of the ICZV point of the connecting rod AB is the intersection of the perpendicular lines to those velocity as depicted in fig. 9.58. Of course the ICZV point of the beam BCE is the fixed rotating point C. After calculating the exact coordinates of the ICZV point,

$[\omega = v/r]$

$$\omega_{AB} = v_A/10.1 = 0.04\pi \text{ rad/s CW}$$

In turn, the angular velocity of the beam BCE is determined to be

$$\omega_{CE} = \frac{v_B}{\sqrt{0.9^2 + 3^2}} = \frac{\omega_{AB} \times 9.92}{\sqrt{0.9^2 + 3^2}} = 0.398 \text{ rad/s CW}$$

Acceleration analysis The acceleration relationship across the rigid connecting rod AB is

$[\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}]$

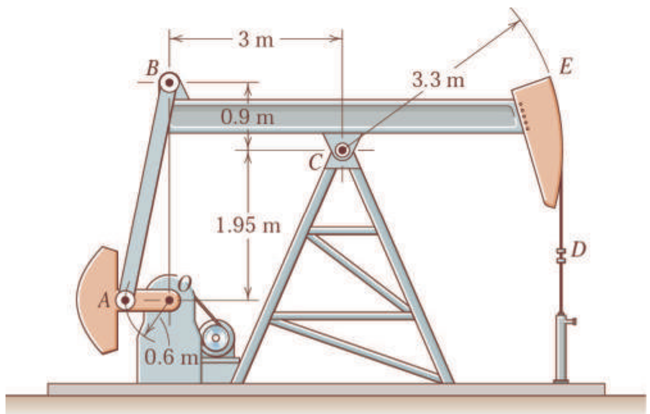


Figure 9.57: Example 9.29 ([4], pp. 382)

The acceleration polygon is sketched in fig. 9.58. Projecting all vectors onto the horizontal direction, the following relation holds:

$$2.632 - 0.046\cos 78.1 - \left(a_{B/A}\right)_t \cos 11.9 - (a_B)_t \sin 16.7 - 0.496\cos 16.7 = 0$$

Similarly, projecting all vectors onto the vertical direction, the following relation holds:

$$-0.046\sin 78.1 + \left(a_{B/A}\right)_t \sin 11.9 - (a_B)_t \cos 16.7 + 0.496\sin 16.7 = 0$$

Solving these two simulatneous equations, the unknowns are determined.

$$\left(a_{B/A}\right)_t = 2.036 \ \ \mathrm{m/s^2}, \qquad (a_B)_t = 0.54 \ \ \mathrm{m/s^2}$$

Consequently, the acceleration of the pumping rod can now be readily determined:

$[\alpha = a_t/r]$ $$\alpha_{CE} = \frac{(a_B)_t}{\sqrt{0.9^2+3^2}} = 0.1724 \ \ \mathrm{rad/s^2} \ \ \mathrm{CW}$$

$$a_D = (a_E)_t = 3.3 \times \alpha_{CE} = 0.569 \ \ \mathrm{m/s^2} \quad \downarrow$$

Example 9.30 ([3], Prob. 5/157) An intermitten-drive mechanism for perforated tape F consists of the link DAB driven by the crank OB. The trace of the motion of the finger at D is shown by the dotted line. Determine the acceleration of D at the instant shown when both OB and CA are horizontal if OB has a constant CW rotational velocity of 120 rev/min.

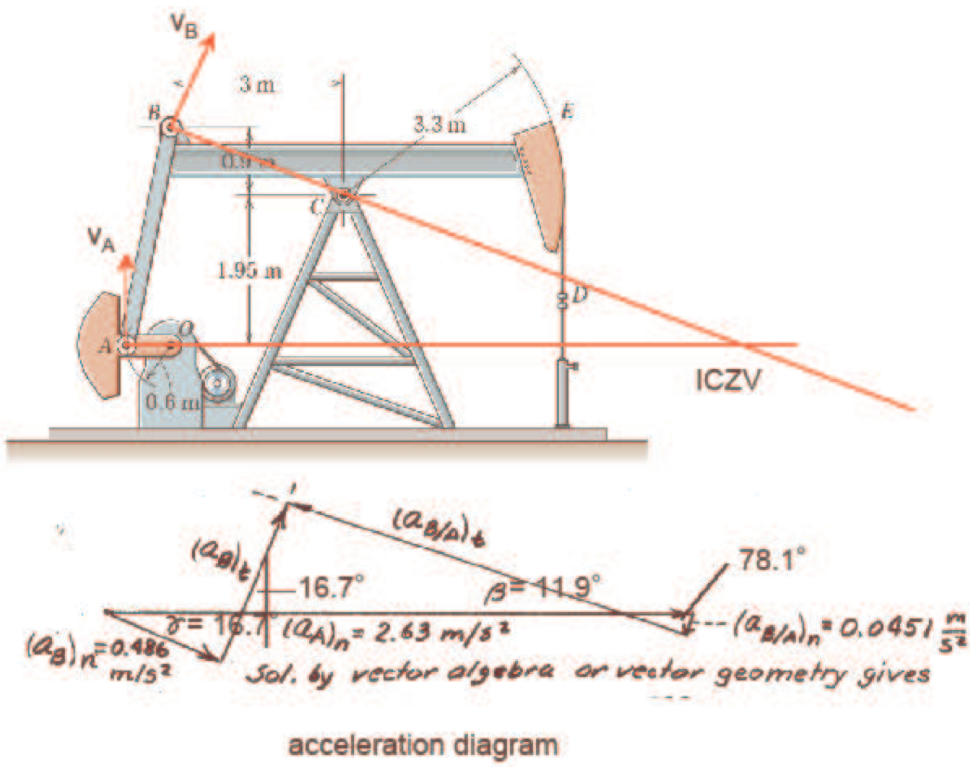


Figure 9.58: Solution to example 9.29

Solution: From the given statement,

$$\omega_{OB} = 120 \times \frac{2\pi}{60} = 4\pi \ \text{ rad/s } \ \text{CW} \ \text{ const}$$

At this particular configuration, the velocity directions of A and B point downward. Therefore, the ICZV is at inf. Consequently, linkage AB translates downward with velocity

$$\omega_{OB} \times \overline{OB} = 0.628 \ \text{ m/s}$$

and the angular velocity of $\overline{AC}$ is

$$\omega_{AC} = \frac{v_A}{\overline{AC}} = 5.0265 \ \text{ rad/s } \ \text{CW}$$

To determine $\mathbf{a}_D$, the acceleration relationship between A and B is analyzed. With this posture and from the velocity information, the total acceleration of B is

$$a_B = \frac{v_B^2}{\overline{OB}} = 7.896 \ \text{ m/s}^2 \quad \leftarrow$$

Similarly, the normal acceleration of A is

$$(a_A)_n = \frac{v_A^2}{\overline{AC}} = 3.158 \ \text{ m/s}^2 \quad \leftarrow$$

Since there is no rotation of link AB at this moment,

$$\left(a_{A/B}\right)_n = 0$$

Sketch the relative acceleration diagram and calculate the length of the unknown sides by geometry analysis, we have

$$\left[\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}\right] \qquad \left(a_{A/B}\right)_t = (7.896 - 3.158) \,/\, \cos 14.4775 = 4.893 \ \text{ m/s}^2$$

Hence,

$$\alpha_{AB} = \frac{\left(a_{A/B}\right)_t}{\overline{AB}} = 24.467 \ \text{ rad/s}^2 \ \text{ CW}$$

Using the acceleration relationship between B and D to determine the acceleration at D. Since the link has no rotation at this moment,

$$\left(a_{D/B}\right)_n = 0$$

$$\left(a_{D/B}\right)_t = \alpha_{AB} \times \overline{BD} = 7.34 \ \text{ m/s}^2$$

$$\left[\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B}\right] \qquad \mathbf{a}_D = (-7.896 + 7.34 \cos 14.4775)\,\mathbf{i} + (7.34 \sin 14.4775)\,\mathbf{j}$$

$$\mathbf{a}_D = -0.789\mathbf{i} + 1.835\mathbf{j} \ \text{ m/s}^2, \qquad a_D = 1.997 \ \text{ m/s}^2$$

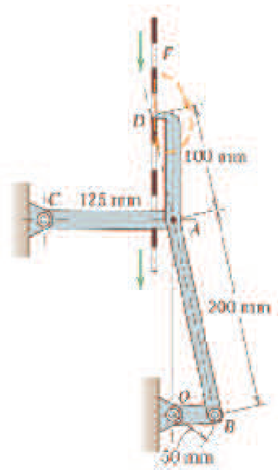


Figure 9.59: Example 9.30 ([3], pp. 394)

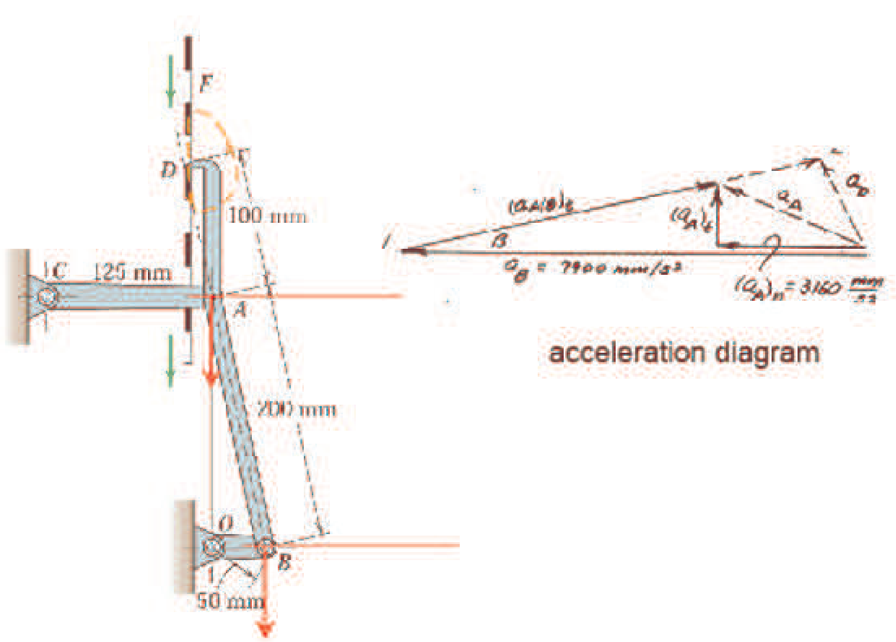


Figure 9.60: Solution to example 9.30

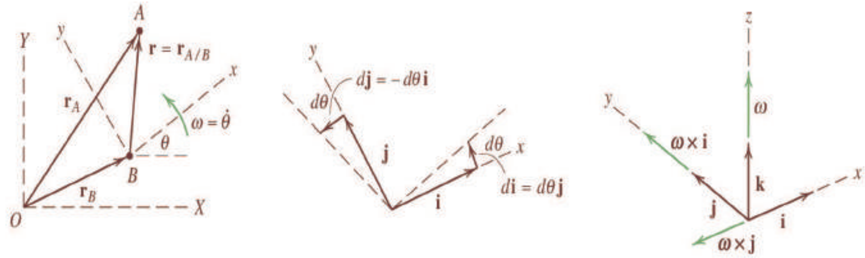


Figure 9.61: Motion relative to the rotating axes frame ([3], pp. 395)

9.7 Motion Relative to Rotating Axes

Up to this point in kinematics analysis, the relative velocity and acceleration, namely $\mathbf{v}_{A/B}$ and $\mathbf{a}_{A/B}$, are measured from the *nonrotating* reference axes. However, there are many situations where the motion is generated within or observed from a system that itself is rotating. In these cases, the analysis is greatly facilitated by the use of the rotating reference axes. One example is the motion of the fluid particle along the curved vane of a rotating pump. Absolute motion of the particle may be thought as being constituted from the addition of the imparting pure rotational motion of the pump blade and the relative motion of the particle to the blade along the constrained curved vane that itself is rotating.

9.7.1 Relative Velocity

Consider two particles A and B moving independently in a plane as depicted in fig. 9.61. Motion of A is observed from a moving reference frame x-y attached to B, which is rotating with an angular velocity $\omega = \dot{\theta}$. From the figure, the positional vectors relationship can be written as

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B} = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j}) \tag{9.21}$$

It is evident that $\mathbf{i}$ and $\mathbf{j}$ are not constant since their directions change. Their rate of changes with respect to time can be determined by investigating the drawing of the unit vector $\mathbf{i}$ and $\mathbf{j}$ rotating by the angle of $d\theta$ for the elapsed time of dt, shown in fig. 9.61. With the definition of the differentiation, the following equations can be derived:

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \times \mathbf{i} = \omega\mathbf{j} \tag{9.22}$$

$$\frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \times \mathbf{j} = -\omega\mathbf{i} \tag{9.23}$$

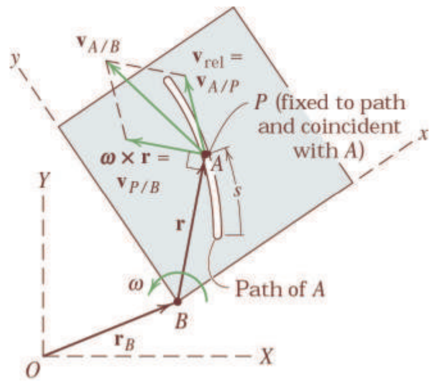


Figure 9.62: Visualization of relative velocity equation using the virtual coincident point P ([3], pp. 396)

Differentiating eq. 9.21 with respect to time and making use of the above relations, the following relative velocity relationship results:

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B} + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{\text{rel}} \tag{9.24}$$

In this equation, velocity of A is observed from the moving reference frame moving with B and rotating with the angular velocity ω. The meaning of each term is as followed. $\mathbf{v}_A$ is the velocity of the particle A and $\mathbf{v}_B$ the velocity of the particle B. An observer at B or *at anywhere* fixed to the rotating frame will *see* A to move with the velocity $\mathbf{v}_{\text{rel}}$. This is contrast to $\mathbf{v}_{A/B}$ which indicates the velocity of A relative to B. $\mathbf{v}_{A/B}$ can be viewed as the velocity of A seen from the *nonrotating* frame moving with B. From eq. 9.24, the relative velocity is

$$\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{\text{rel}}$$

Therefore, $\boldsymbol{\omega} \times \mathbf{r}_{A/B}$ is the difference between the velocity of A observed from the nonrotating and the rotating frame. In other words, it is the unconscious relative velocity between the target and the observer caused by his rotational motion.

If we use the nonrotating frame, there will be no $\boldsymbol{\omega} \times \mathbf{r}_{A/B}$ term. The equation degenerates to eq. 7.30 explained in chapter 7. This makes $\mathbf{v}_{\text{rel}} = \mathbf{v}_{A/B}$, which means the velocity seen by the observer is the velocity of A realtive to B. If B coincides with A, $\mathbf{r}_{A/B} = 0$. This also causes $\mathbf{v}_{\text{rel}} = \mathbf{v}_{A/B}$, which means the velocity seen by the observer is the velocity of A relative to B, even the observer is rotating.

Another way to comprehend the relative velocity equation using the rotating frame is to visualize the situation shown in fig. 9.62. There are two particles A and B which may be on different rigid bodies. Imagine there is a rotating plate on which the particle B is situated. *On this plate*, imagine the virtual point P currently coincident with A. As a result, the relationship between $\mathbf{v}_P$ and $\mathbf{v}_B$ is indicated by

$$\mathbf{v}_P = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{P/B} = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

Since point A is on the different body, $\mathbf{v}_A \neq \mathbf{v}_P$. From the above relation, it can be concluded that the velocity of A as seen from P is $\mathbf{v}_{\text{rel}}$. In fact, the observer at any location fixed to the rotating plane will see A to be moving with the velocity $\mathbf{v}_{\text{rel}}$. For this particular illustration, the direction of $\mathbf{v}_{\text{rel}}$ is tangent to the path (slot) fixed in the rotating plate. That is, the virtual slot is constructed to be coincided with the trajectory seen by the observer fixed to the rotating plate. Note that *it is not the absolute path of* A (which must be measured by the fixed observer).

From the illustration in fig. 9.62, the magnitude of $\mathbf{v}_{\text{rel}}$, or the relative speed, is $\dot{s}$. Since P is the point fixed to the moving plate that is instantaneously coincident with A, it can be concluded that $\mathbf{v}_{\text{rel}} = \mathbf{v}_{A/P}$. That is, the velocity of A seen by the rotating observer B or P is the same, and it is equal to the velocity of A relative to P. Be caution that *it is not the same as the velocity of A relative to B*.

To summarize, observers moving with different velocities (different $\mathbf{v}_B$) on the same rotating x-y frame see the target moving with the same velocity of $\mathbf{v}_{\text{rel}}$. Observers moving with the same velocity, but are on different rotating x-y frames (different $\boldsymbol{\omega}$), see the target moving with different velocity $\mathbf{v}_{\text{rel}}$. Non-rotating observers will see the resultant of circular motion plus the relative velocity. The following relative velocity equations summarize the relative motion of point A and B from different point of views.

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{\text{rel}} \quad (9.25)$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \quad (9.26)$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{P/B} + \mathbf{v}_{A/P} \quad (9.27)$$

$$\mathbf{v}_A = \mathbf{v}_P + \mathbf{v}_{A/P} \quad (9.28)$$

9.7.2 Vector Differentiation

The change of a vector with respect to time as seen from a general reference frame depends on the intrinsic change of the vector itself and the change of the vector induced by the motion of the reference frame, whether it be the translation or the rotation. Here, the interested frame is constrained not to translate, but to rotate around its origin.

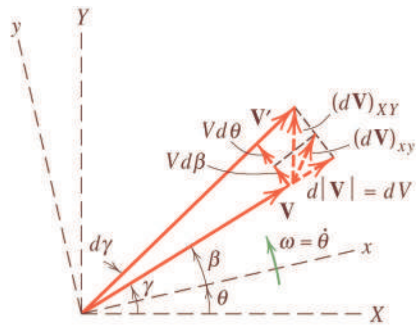


Figure 9.63: Time rate of change of a vector **V** ([3], pp. 397)

An arbitrary vector $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j}$ has the time derivative

$$\left(\frac{d\mathbf{V}}{dt}\right)_{\text{XY}} = \left(\dot{V}_x\mathbf{i} + \dot{V}_y\mathbf{j}\right) + \left(V_x\dot{\mathbf{i}} + V_y\dot{\mathbf{j}}\right) \tag{9.29}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{\text{XY}} = \left(\frac{d\mathbf{V}}{dt}\right)_{\text{xy}} + \boldsymbol{\omega} \times \mathbf{V} \tag{9.30}$$

This expression means the time derivative of **V** measured in the fixed frame (the total time derivative) is equal to the time derivative of **V** as measured in the rotating frame plus the compensation due to rotation of the reference frame.

More insight can be seen from the vector diagram in fig. 9.63. Vector **V** changes in both direction and magnitude to $\mathbf{V}'$. In the figure, x-y frame changes by rotating with the angular velocity $\boldsymbol{\omega}$ while X-Y frame is fixed. During the time interval dt, *the observer in rotating x-y frame see the change in magnitude of* **V**, dV, *plus the change in direction,* $V\,d\beta$, *due to the relative rotation of* **V** *to* x-y. The change that the rotating observer recognized is called $(d\mathbf{V})_{xy}$. *What it does not notice is the rotation of* **V** *induced by the rotation of* x-y, $V\,d\theta$. Imagine that **V** is fixed to x-y. Therefore, its direction changes by the rotation of x-y frame, which is not known to the observer rotating together.

This fundamental relationship can be used to derive the relative acceleration equation from the relative velocity equation.

9.7.3 Relative Acceleration

The relative acceleration relationship may be obtained by directly differentiating the relative velocity equation 9.24. Accordingly,

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{A/B} + \dot{\mathbf{v}}_{\text{rel}}$$

With the use of eq. 9.30, the differentiation terms above can be further exemplified as

$$\dot{\mathbf{r}}_{A/B} = \mathbf{v}_{\text{rel}} + \boldsymbol{\omega}\times\mathbf{r}_{A/B} \tag{9.31}$$

$$\dot{\mathbf{v}}_{\text{rel}} = \mathbf{a}_{\text{rel}} + \boldsymbol{\omega}\times\mathbf{v}_{\text{rel}} \tag{9.32}$$

$$\dot{\boldsymbol{\omega}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{xy}} + \boldsymbol{\omega}\times\boldsymbol{\omega}$$

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{XY}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{xy}} \tag{9.33}$$

The last equation implies the angular acceleration seen in the rotating frame is the absolute angular acceleration because the vector of the angular velocity aligns with the angular velocity of the observing frame. Substituting these terms into the acceleration relationship, the relative acceleration equation employing the rotating frame results:

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}_{A/B}\right) + 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{9.34}$$

In this equation, the acceleration of A is observed from the moving reference frame moving with B and rotating with the angular velocity and acceleration of ω and $\dot{\omega}$. The meaning of each term is as followed. $\mathbf{a}_A$ and $\mathbf{a}_B$ are the acceleration of the particle A and B, respectively. An observer at B or *at anywhere* fixed to the rotating frame will *see* A to move with the acceleration $\mathbf{a}_{\text{rel}}$. This is contrast to $\mathbf{a}_{A/B}$ which indicates the acceleration of A relative to B. $\mathbf{a}_{A/B}$ can be viewed as the acceleration of A seen from the *nonrotating* frame moving with B. From eq. 9.34, the relative acceleration is

$$\mathbf{a}_{A/B} = \dot{\boldsymbol{\omega}}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}_{A/B}\right) + 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

Therefore, $\dot{\boldsymbol{\omega}}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}_{A/B}\right) + 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}}$ is the difference between the acceleration of A observed from the nonrotating and the rotating frame. It is the unconscious relative acceleration between the target and the observer caused by his rotational motion. For the nonrotating frame, the equation degenerates to eq. 7.32 explained in chapter 7. This makes $\mathbf{a}_{\text{rel}} = \mathbf{a}_{A/B}$, which means the acceleration seen by the observer is the acceleration of A relative to B. If B coincides with A, $\mathbf{r}_{A/B} = 0$. This makes $\mathbf{a}_{\text{rel}} = \mathbf{a}_{A/B} - 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}}$, indicating that the acceleration seen by the rotating observer may not be the relative acceleration of A relative to B, even they are coincident. Note that $\mathbf{a}_{\text{rel}} \neq \dot{\mathbf{v}}_{\text{rel}}$.

Another way to comprehend the relative acceleration equation using the rotating frame is to visualize the situation shown in fig. 9.64. There are two particles A and B which may be on different rigid bodies. Imagine there is a rotating plate on which the particle B is situated. *On this plate*, imagine the virtual point P currently coincident with A. Therefore, P is seen to perform circular motion

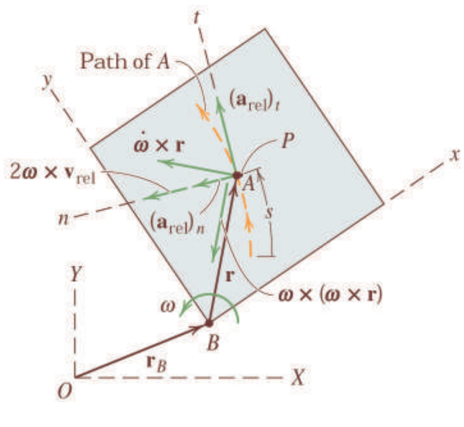


Figure 9.64: Visualization of relative acceleration equation using the virtual co-incident point P ([3], pp. 398)

about the nonrotating observer at B. As a result, the relationship between $\mathbf{a}_P$ and $\mathbf{a}_B$ is determined by

$$\mathbf{a}_P = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}_{A/B}\right)$$

Since point A is on the different body, $\mathbf{a}_A \neq \mathbf{a}_P$. Unlike the velocity, however, the acceleration difference $\mathbf{a}_A - \mathbf{a}_P$ is not the acceleration of A seen from P. The velocity of A as seen from P, $\mathbf{v}_{\text{rel}}$, changes due to the vector itself ($\mathbf{a}_{\text{rel}}$) and due to the induced motion of the rotating frame ($\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}}$). If the observer is on the frame rotating with the rigid body, he will not be able to observe the induced motion. In other words, *any* observer in the rotating frame will see point A moving with the acceleration $\mathbf{a}_{\text{rel}}$. Generally, $\mathbf{a}_{\text{rel}}$ has components in both the normal and tangential directions to the path (slot) fixed in the rotating plate, for the virtual slot is constructed to be coincided with the trajectory seen by the observer fixed in the rotating plate. Note that *it is not the absolute path of A* (which must be measured by the fixed observer).

One might tempt to think that, apart from the centrifugal and the tangential accelerations of the coincident virtual point P, $\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}}$ is the acceleration the rotating observer does not perceive. This is incorrect, though. Change of the relative position vector, $\mathbf{r}_{A/B}$, itself, denoted by $\mathbf{v}_{\text{rel}}$, accounts for the other unobservable acceleration $\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}}$. Combining these two terms results in the Coriolis acceleration $2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}}$, named after the scientist name who discovered this missing acceleration. For the Coriolis acceleration to exist, both $\boldsymbol{\omega}$ and $\mathbf{v}_{\text{rel}}$ must be nonzero. In particular, particle A must be *moving on the rotating platform.*

From the illustration in fig. 9.64, $\mathbf{a}_{\text{rel}}$ is the acceleration required to move along the relative path (slot). Had the path been known, the n-t coordinate system is the most suitable description. Accordingly, $\mathbf{a}_{\text{rel}}$ can be decomposed into the normal and the tangent direction for which

$$\begin{aligned}(a_{\text{rel}})_t &= \ddot{s}\\ (a_{\text{rel}})_n &= v_{\text{rel}}^2/\rho\end{aligned}$$

In summary, $\mathbf{a}_{\text{rel}}$ is the change of $\mathbf{v}_{\text{rel}}$ observed from any location in the *rotating frame*. This is the acceleration of A seen by any rotating observer. It is not equal to $\dot{\mathbf{v}}_{\text{rel}}\ (= \boldsymbol{\omega}\times\mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}})$ which is just the change of $\mathbf{v}_{\text{rel}}$ observed in the *nonrotating frame*. Also, it is not equal to $\mathbf{a}_{A/P}\ (= 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}})$ which is the acceleration of A seen by the coincident point P in the *nonrotating frame*. If the observer is not coincident to A, he will see the resultant of normal, tangential (portion of the relative motion which makes up the relative circular motion), Coriolis, and relative acceleration (the rest of the relative motion by virtual slot on the rotating frame). The following relative acceleration equations summarize the relative motion of point A and B from different point of views.

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}_{A/B}\right) + 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{9.35}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} \tag{9.36}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{P/B} + \mathbf{a}_{A/P} \tag{9.37}$$

$$\mathbf{a}_A = \mathbf{a}_P + \mathbf{a}_{A/P} \tag{9.38}$$

Example 9.31 ([3], Prob. 5/183) The crank OA revolves clockwise with a constant angular velocity of 10 rad/s within a limited arc of its motion. For the position $\theta = 30°$, determine the angular velocity of the slotted link CB and the acceleration of A as measured relative to the slot in CB.

Solution: Typically, the velocity information must be evaluated before calculating the acceleration because of the appearance of the velocity terms in the acceleration equation. Motion of the crank OA is transmitted to the slot CB through the pin A. Therefore, the following relative velocity equation is set up with the helpful velocity diagram shown in fig. 9.66:

$$\left[\mathbf{v}_A = \mathbf{v}_P + \mathbf{v}_{A/P}\right]$$

From the given data, $v_A = 0.2\times 10 = 2$ m/s. Completing the velocity diagram in fig. 9.66, the pertinent velocities can be determined as

$$v_P = 2\cos 30 = 2\times 0.2\cos 30\times\omega_{CB},\quad \omega_{CB} = 5\ \text{ rad/s CW}$$

$$v_{A/P} = v_{\text{rel}} = 2\sin 30 = 1\ \text{ m/s}$$

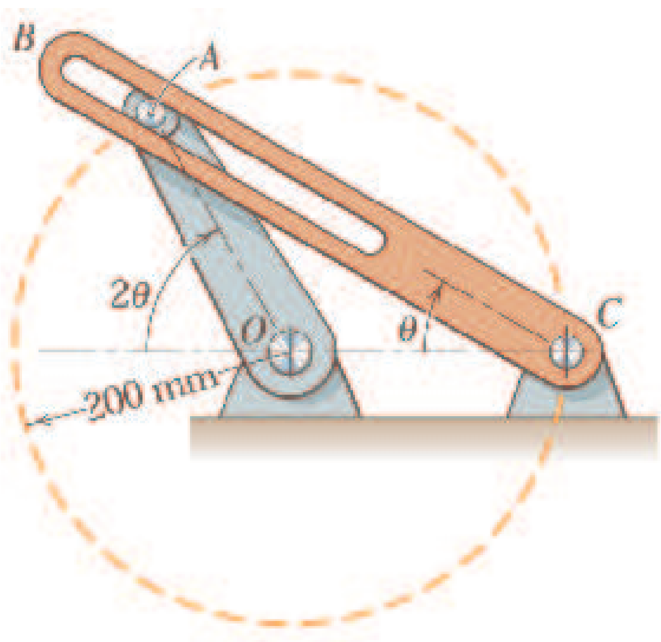


Figure 9.65: Example 9.31 ([3], pp. 410)

Similarly, apply the relative acceleration equation between the point A on the crank and the fixed point C on the slot linkage. If the observer is at C and is rotating along with the slot, he would see A to be moving along the straight slot. Therefore,

$$\left[\mathbf{a}_A = \mathbf{a}_C + \boldsymbol{\omega}_{CB} \times \boldsymbol{\omega}_{CB} \times \mathbf{r}_{A/C} + \dot{\boldsymbol{\omega}}_{CB} \times \mathbf{r}_{A/C} + 2\boldsymbol{\omega}_{CB} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}\right]$$

From the velocity analysis,

$$\left|\boldsymbol{\omega}_{CB} \times \boldsymbol{\omega}_{CB} \times \mathbf{r}_{A/C}\right| = 8.66 \ \ \text{m/s}^2$$

$$\left|2\boldsymbol{\omega}_{CB} \times \mathbf{v}_{\text{rel}}\right| = 10 \ \ \text{m/s}^2$$

$$a_A = v_A^2/\overline{OA} = 20 \ \ \text{m/s}^2$$

Construct the acceleration diagram as depicted in fig. 9.66 and perform the geometrical analysis, the remaining acceleration can be determined.

$$a_{\text{rel}} = 20\cos 30 - 8.66 = 8.66 \ \ \text{m/s}^2 \ \ \text{along the slot towards } C$$

$$\left|\dot{\boldsymbol{\omega}}_{CB} \times \mathbf{r}_{A/C}\right| = 20\cos 60 - 10 = 0, \quad \dot{\omega}_{CB} = 0 \ \ \text{rad/s}^2$$

Example 9.32 ([4], Prob. 5/175) Determine the angular acceleration α_2 of wheel C for the instant when $\theta = 20°$. Wheel A has a constant clockwise angular

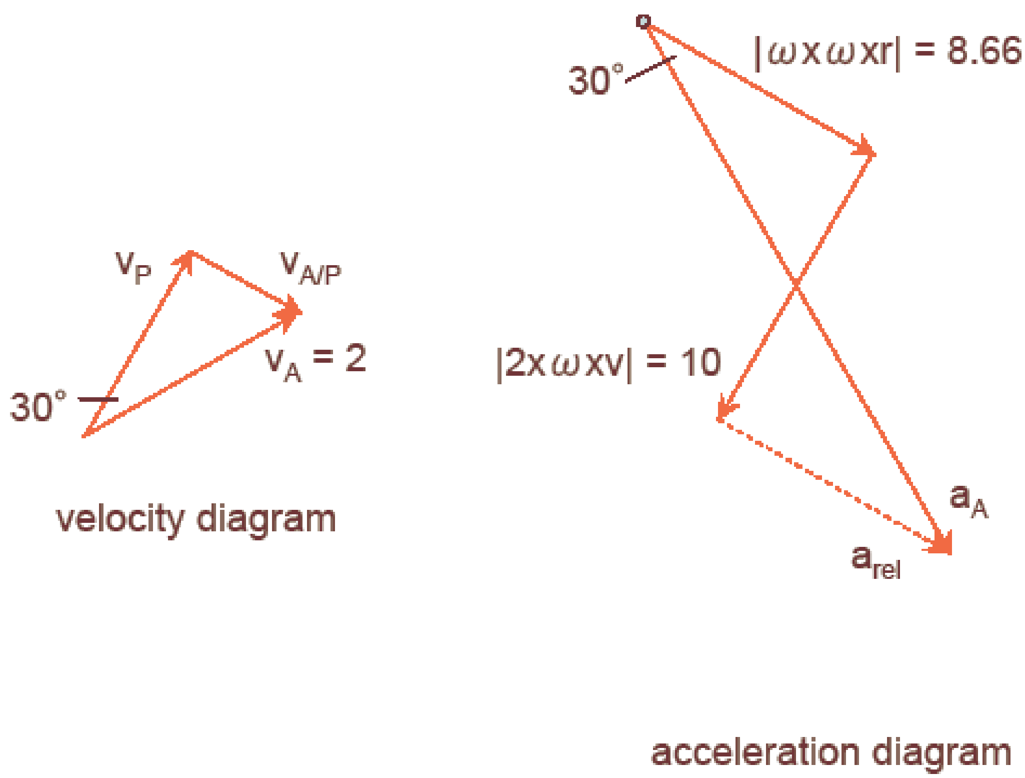


Figure 9.66: Solution to example 9.31

velocity of 2 rad/s.

Solution: Motion of the wheel A is transmitted to the wheel C of the Geneva intermitten mechanism through the pin P and the slots. Therefore the relative motion equations should be set so that the motion of the coincident points on the wheel A and C are involved.

Geometry analysis First, the geometric parameters at this instant are determined. Consider the triangle O_1PO_2 when $\theta = 20°$ shown in fig. 9.68.

$$\tan\alpha = \frac{\frac{200}{\sqrt{2}}\sin 20}{200 - \frac{200}{\sqrt{2}}\cos 20}, \quad \alpha = 35.78°$$

$$\overline{PO_2}^2 = \left(\frac{200}{\sqrt{2}}\sin 20\right)^2 + \left(200 - \frac{200}{\sqrt{2}}\cos 20\right)^2, \quad \overline{PO_2} = 82.7 \text{ mm}$$

Velocity analysis Next, the velocity relationship of the pin-coincident points A on the wheel A and P on the wheel C is formulated.

$$[\mathbf{v}_A = \mathbf{v}_P + \mathbf{v}_{\text{rel}}]$$

Since point A and P move on the circular path,

$$v_A = 2 \times 0.2/\sqrt{2} = 0.283 \text{ m/s}$$

Draw the velocity diagram and perform the geometrical analysis to unveil $\mathbf{v}_P$ and $\mathbf{v}_{\text{rel}}$.

$$v_P = v_A \cos 55.78 = 0.159 = \omega_2 \times 0.0827, \quad \omega_2 = 1.923 \text{ rad/s CCW}$$

$$v_{\text{rel}} = v_{A/P} = v_A \sin 55.78 = 0.234 \text{ m/s}$$

Acceleration analysis Now there are enough information to perform the acceleration analysis. The acceleration of point A observed from the person at O_2 with the eyesight rotating with the wheel C abides by the following equation:

$$[\mathbf{a}_A = \mathbf{a}_{O_2} + \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_2 \times \mathbf{r}_{A/O} + \dot{\boldsymbol{\omega}}_2 \times \mathbf{r}_{A/O} + 2\boldsymbol{\omega}_2 \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}]$$

Since the path of A seen on the rotating wheel C is along the slot, the direction of $\mathbf{a}_{\text{rel}}$ is parallel to the slot. Considering the unknowns in the above equation, they are the magnitudes of $\dot{\boldsymbol{\omega}}_2 \times \mathbf{r}_{A/O}$ and $\mathbf{a}_{\text{rel}}$ for the two scalar equations to solve. See the acceleration diagram in fig. 9.68. The magnitudes of the known terms are

$$a_{O_2} = 2 \times 2 \times 0.2/\sqrt{2} = 0.5656 \text{ m/s}^2$$

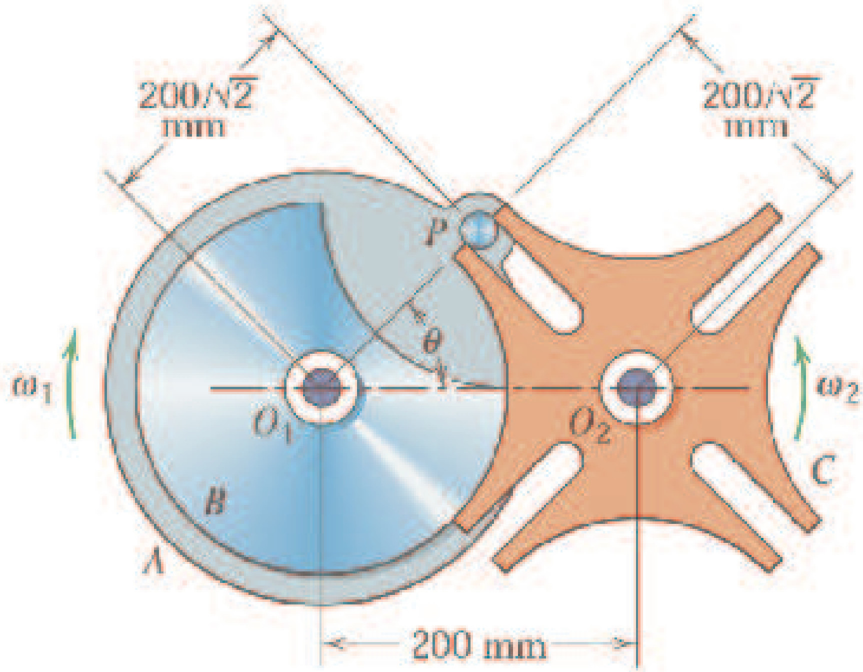


Figure 9.67: Example 9.32 ([4], pp. 397)

$$\left|\boldsymbol{\omega}_2 \times \boldsymbol{\omega}_2 \times \mathbf{r}_{A/O}\right| = 0.306 \text{ m/s}^2$$

$$\left|2\boldsymbol{\omega}_2 \times \mathbf{v}_{\text{rel}}\right| = 0.9 \text{ m/s}^2$$

recognizing the fact that point A is moving on the circular path at constant angular velocity.

With the acceleration diagram, projecting the sides of the vector polygon onto the $\dot{\boldsymbol{\omega}}_2 \times \mathbf{r}_{A/O}$ direction, the following relationship can be written:

$$\left|\dot{\boldsymbol{\omega}}_2 \times \mathbf{r}_{A/O}\right| = 0.9 + 0.5656 \cos 34.22$$

Therefore the angular acceleration of the wheel C is

$$\dot{\omega}_2 = \left|\dot{\boldsymbol{\omega}}_2 \times \mathbf{r}_{A/O}\right| / \left|\mathbf{r}_{A/O}\right| = 16.54 \text{ rad/s}^2 \text{ CCW}$$

Example 9.33 ([3], Prob. 5/186) The space shuttle A is in an equilateral circular orbit of 240 km altitude and is moving from west to east. Determine the velocity and acceleration which it appears to have to an observer B fixed to and rotating with the earth at the equator as the shuttle passed overhead. Use $R = 6378$ km for the radius of the earth. Also use Fig. 1/1 for the appropriate value of g and carry out your calculation to 4-figure accuracy.

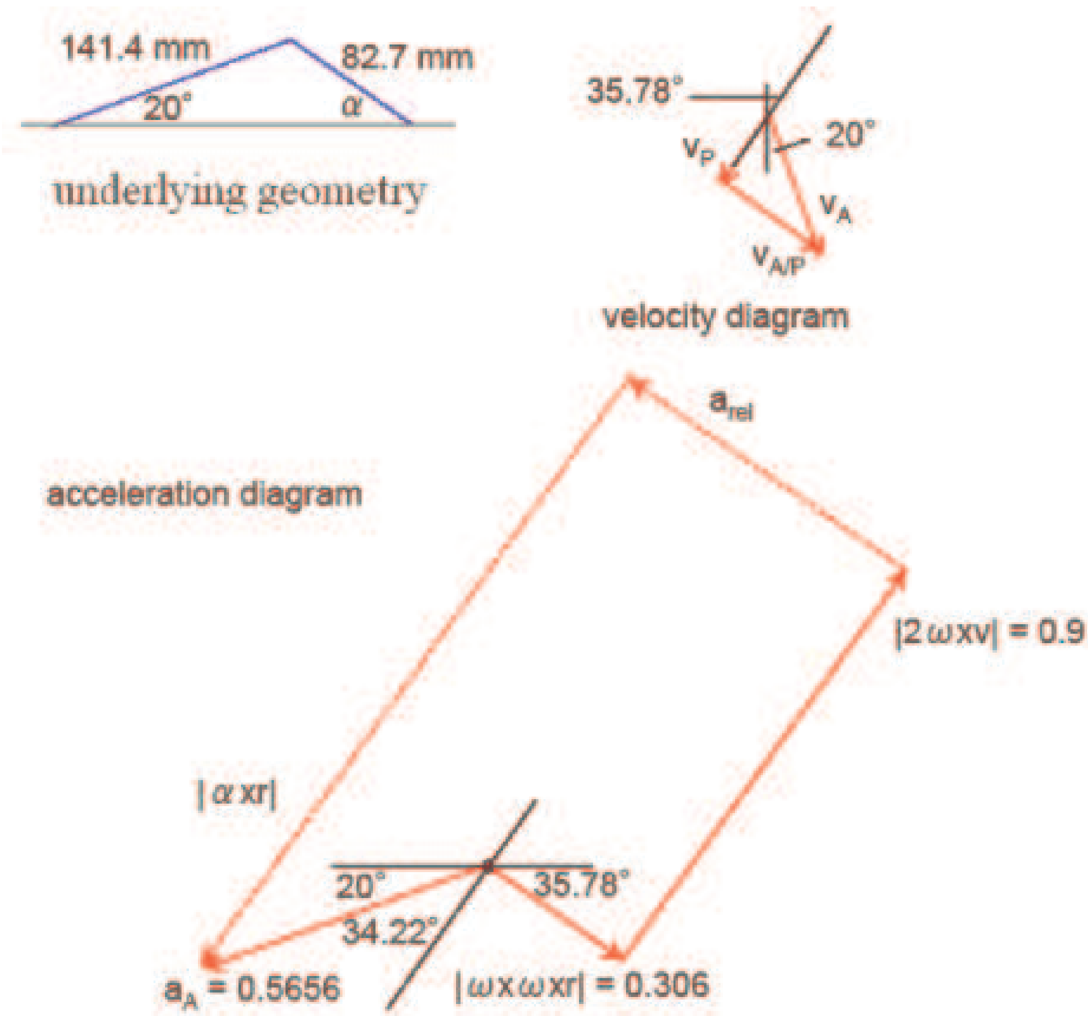


Figure 9.68: Solution to example 9.32

Solution: The observer B is on the rotating earth which leads us to apply the motion equations relative to the rotating axes. Motion of the space shuttle and the observer are then related as follow:

$$\left[\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega}_e \times \mathbf{r}_{A/B} + \mathbf{v}_{\text{rel}}\right]$$

From the appendix and assuming the axis of the rotation of the earth is normal to the equilateral plane, $\omega_e = 0.7292 \times 10^{-4}$ rad/s constant CCW. Therefore,

$$v_B = \omega_e \times 6378 \times 10^3 = 465.084 \;\; \text{m/s} \quad \leftarrow$$

To determine the velocity of the space shuttle, recall the circular-orbit shuttle has the normal acceleration towards the center of the earth of which its value is

$$(a_A)_n = g\left(\frac{R}{R+h}\right)^2 = \frac{v_A^2}{R+h}$$

Consulting the appendix for the accurate value of the gravitational constant, $g = 9.814 \;\; \text{m/s}^2$. Thus

$$v_A = 7766.79 \;\; \text{m/s} \quad \leftarrow$$

Applying the relative velocity equation above, the velocity seen by the observer B, $\mathbf{v}_{\text{rel}}$, can now be determined.

$$-7766.79\mathbf{i} = -465.084\mathbf{i} - 0.7292 \times 10^{-4} \times 240 \times 10^3 + \mathbf{v}_{\text{rel}}$$

$$v_{\text{rel}} = 7284.205 \;\; \text{m/s} = 26223 \;\; \text{km/h} \quad \leftarrow$$

Consequently, the acceleration of the space shuttle seen by the observer can be determined from the acceleration equation relative to the rotating axes as

$$\left[\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\omega}_e \times \boldsymbol{\omega}_e \times \mathbf{r}_{A/B} + \dot{\boldsymbol{\omega}}_e \times \mathbf{r}_{A/B} + 2\boldsymbol{\omega}_e \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}\right]$$

Let us first count the unknowns in the equation. From the velocity analysis, $(\mathbf{a}_A)_n$ has been determined. However, the magnitude of $(\mathbf{a}_A)_t$ is the unknown. The other unknown is the acceleration seen by the observer, $\mathbf{a}_{\text{rel}}$. Hence there are three scalar unknowns in this two-dimensional vector equation for which not all of them may be solved.

Assume that the shuttle orbits with the constant velocity magnitude, $(\mathbf{a}_A)_t = \mathbf{0}$ and so $\mathbf{a}_A = (\mathbf{a}_A)_n$. As a result, only $\mathbf{a}_{\text{rel}}$ provides two scalar unknowns which can be solved from the equation. Computing the values of each term in the acceleration equation,

$$\begin{aligned} -9.81\left(\frac{6378}{6378+240}\right)^2\mathbf{j} &= -\left(0.7292 \times 10^{-4}\right)^2 \times 6378 \times 10^3\mathbf{j} \\ &- \left(0.7292 \times 10^{-4}\right)^2 \times 240 \times 10^3\mathbf{j} - 2 \times 0.7292 \times 10^{-4} \times 7284.205\mathbf{j} + \mathbf{a}_{\text{rel}} \end{aligned}$$

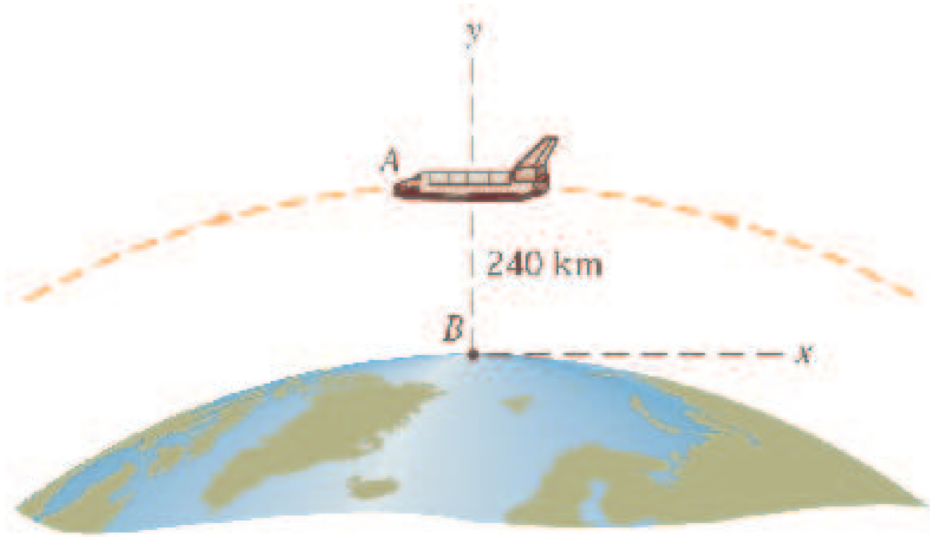


Figure 9.69: Example 9.33 ([3], pp. 410)

Here the angular velocity of the earth is presumed to be constant as well which makes $\mathbf{a}_B = (\mathbf{a}_B)_n$. Solving the equation for the acceleration of the space shuttle seen by the observer B,

$$a_{\text{rel}} = 8.018 \ \ \text{m/s}^2 \quad \downarrow$$

Example 9.34 ([3], Prob. 5/179) The figure shows the vanes of a centrifugal pump impeller which turns with a constant clockwise speed of 200 rev/min. The fluid particles are observed to have an absolute velocity whose component in the r-direction is 3 m/s at discharge from the vane. Furthermore, the magnitude of the velocity of the particles measured relative to the vane is increasing at the rate of 24 m/s^2 just before they leave the vane. Determine the magnitude of the total acceleration of a fluid particle an instant before it leaves the impeller. The radius of curvature ρ of the vane at its end is 200 mm.

Solution: Let the observer P rotating with the impeller be coincident to the fluid particles about to leave the vane at this instant. The velocity of the fluid particles seen by this observer is determined by the following relationship;

Velocity analysis

$$\left[\mathbf{v}_A = \mathbf{v}_P + \boldsymbol{\omega} \times \mathbf{r}_{A/P} + \mathbf{v}_{\text{rel}}\right]$$

With the help of the velocity diagram shown in fig. 9.71 and expressing the

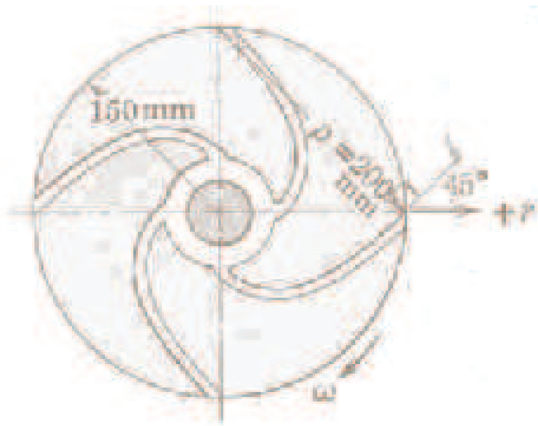


Figure 9.70: Example 9.34 ([3], pp. 409)

vector quantities in r-θ coordinates, the equation can be written specifically as

$$(v_A)_\theta \, \mathbf{e}_\theta + 3\mathbf{e}_r = 200 \times \frac{2\pi}{60} \times 0.15\mathbf{e}_\theta + v_{\text{rel}} \left(\cos 45\mathbf{e}_r - \sin 45\mathbf{e}_\theta\right)$$

Consequently,

$$v_{\text{rel}} = 3\sqrt{2} \;\; \text{m/s} \;\; \text{and} \;\; (v_A)_\theta = (\pi - 3) \;\;\; \text{m/s}$$

Acceleration analysis Absolute acceleration of the fluid particles can be readily determined from the observer acceleration and the apparent acceleration as follow.

$$\left[\mathbf{a}_A = \mathbf{a}_P + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}_{A/P} + \dot{\boldsymbol{\omega}}\times\mathbf{r}_{A/P} + 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}\right]$$

$$\begin{aligned} \mathbf{a}_A &= -0.15 \times \left(\frac{20\pi}{3}\right)^2 \mathbf{e}_r + 2 \times \frac{20\pi}{3} \times 3\sqrt{2} \left(\cos 45\mathbf{e}_r + \sin 45\mathbf{e}_\theta\right) \\ &\quad +24 \left(\cos 45\mathbf{e}_r - \sin 45\mathbf{e}_\theta\right) + \frac{(3\sqrt{2})^2}{0.2} \left(-\cos 45\mathbf{e}_r - \sin 45\mathbf{e}_\theta\right) \\ &= 13.187\mathbf{e}_r + 45.04\mathbf{e}_\theta \end{aligned}$$

$$a_A = 46.93 \;\; \text{m/s}^2$$

Example 9.35 ([4], Prob. 5/176) The mechanism shown is a device to produce high torque in the shaft at O. The gear unit, pivoted at C, turns the right-handed screw at a constant speed $N = 100$ rev/min in the direction shown which advances the threaded collar at A along the screw toward C. Determine

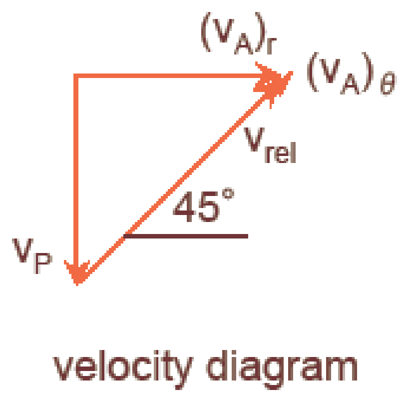


Figure 9.71: Solution to example 9.34

the time rate of change $\dot{\boldsymbol{\omega}}_{AO}$ of the angular velocity of AO as it passes the vertical position shown. The screw has 3 single threads per centimeter of length.

Solution: Rotation of the screw causes the relative translation of the collar A. With the given thread specification, three turns advances the collar by 1 cm. Let A be the point on the collar and P be the coincident point on the screw. Since the screw rotates at a constant speed 100 rev/min,

$$v_{A/P} = \frac{100}{3} \text{ cm/min} = \frac{1}{180} \text{ m/s}$$

in the direction along the screw towards C.

Velocity analysis Using the velocity relationship and the velocity diagram in fig. 9.73, the absolute velocity of the collar and the links' angular velocities can be determined.

$[\mathbf{v}_A = \mathbf{v}_P + \mathbf{v}_{\text{rel}}]$

$$v_A = \frac{1/180}{\cos 29.745} = \omega_{OA} \times 0.2, \quad \omega_{OA} = 0.032 \text{ rad/s CCW}$$

$$v_P = v_{\text{rel}} \tan 29.745 = 0.0032 \text{ m/s}, \quad \omega_{PC} = v_P/\overline{CP} = 0.00788 \text{ rad/s CCW}$$

Acceleration analysis Establishing the acceleration relationship between the point A and P and acknowledging that

$$\mathbf{r}_{A/P} = \mathbf{0},$$

the kinematical relationship simplifies to

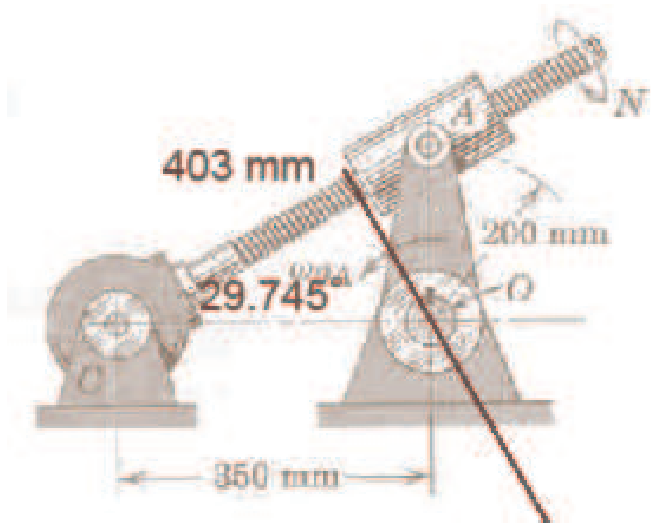


Figure 9.72: Example 9.35 ([4], pp. 398)

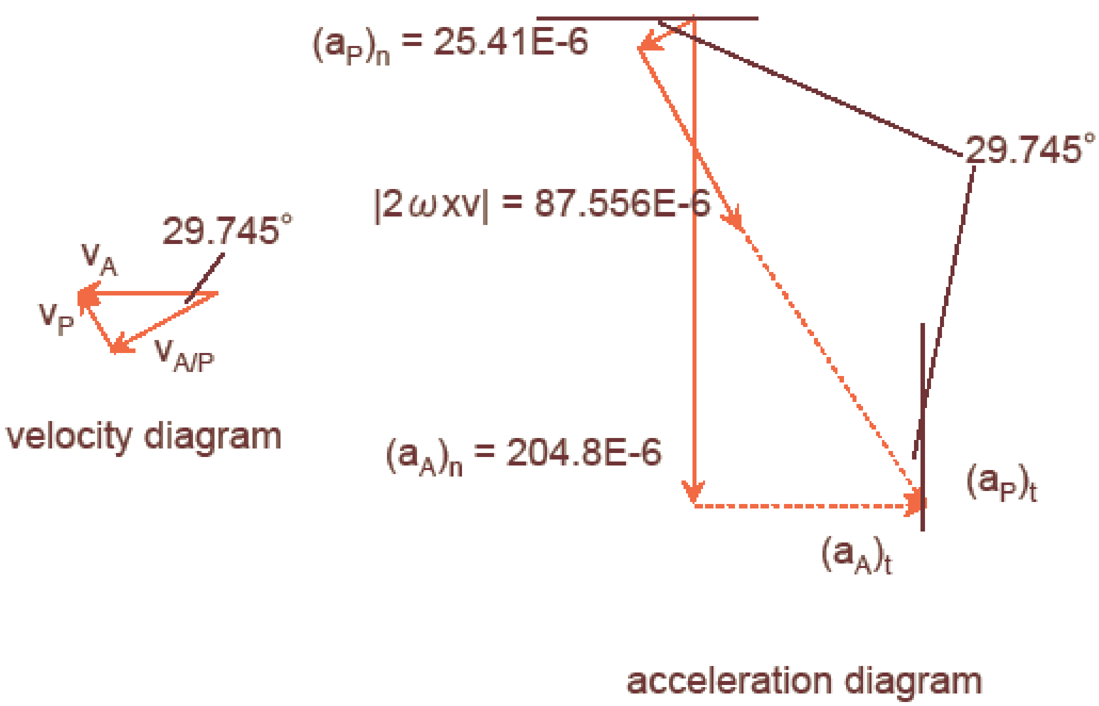


Figure 9.73: Solution to example 9.35

$\left[\mathbf{a}_A = \mathbf{a}_P + 2\boldsymbol{\omega}_{PC} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}\right]$

The corresponding acceleration diagram is drawn in fig. 9.73. For this case, $\mathbf{a}_{\text{rel}} = \mathbf{0}$ because the screw turns at a constant rate. The centrifugal acceleration component of $\mathbf{a}_A$ and $\mathbf{a}_P$ can be calculated as

$[a_n = r\omega^2]$
$$(\mathbf{a}_A)_n = 0.2 \times \omega_{OA}^2 = 204.8 \times 10^{-6} \ \ \text{m/s}^2$$

$$(\mathbf{a}_P)_n = \overline{AC} \times \omega_{PC}^2 = 25.41 \times 10^{-6} \ \ \text{m/s}^2$$

From the velocity analysis, the Coriolis acceleration is

$$\left|2\boldsymbol{\omega}_{PC} \times \mathbf{v}_{\text{rel}}\right| = 87.556 \times 10^{-6} \ \ \text{m/s}^2$$

To determine the unknown tangential acceleration component of $\mathbf{a}_A$ and $\mathbf{a}_P$, the closed loop acceleration polygon are projected onto the horizontal and the vertical direction. In the vertical direction,

$$204.8 \times 10^{-6} = 25.41 \times 10^{-6} \sin 29.745° + 87.556 \times 10^{-6} \cos 29.745° + (\mathbf{a}_P)_t \cos 29.745°$$

$$(\mathbf{a}_P)_t = 133.8 \times 10^{-6} \ \ \text{m/s}^2$$

In the horizontal direction,

$$(\mathbf{a}_A)_t = \dot{\omega}_{OA} \times 0.2 = -25.41 \times 10^{-6} \cos 29.745° + (87.556 + 133.8) \times 10^{-6} \sin 29.745°$$

$$\dot{\omega}_{OA} = 438.8 \times 10^{-6} \ \ \text{rad/s} \ \ \text{CW}$$

Example 9.36 ([3], Prob. 5/185) Determine the angular acceleration of link EC in the position shown, where $\omega = \dot{\beta} = 2$ rad/s and $\ddot{\beta} = 6\text{rad/s}^2$ when $\theta = \beta = 60°$. Pin A is fixed to link EC. The circular slot in link DO has a radius of curvature of 150 mm. In the position shown, the tangent to the slot at the point of contact is parallel to AO.

Solution: Let P be the virtual point on the link OD that is coincident with the pin A. Therefore

$$\mathbf{r}_{A/P} = \mathbf{0}$$

Velocity analysis With the current geometry of the mechanism, the velocity relationship and its diagram (fig. 9.75) of point A and P can be determined as

$\left[\mathbf{v}_A = \mathbf{v}_P + \mathbf{v}_{\text{rel}}\right]$

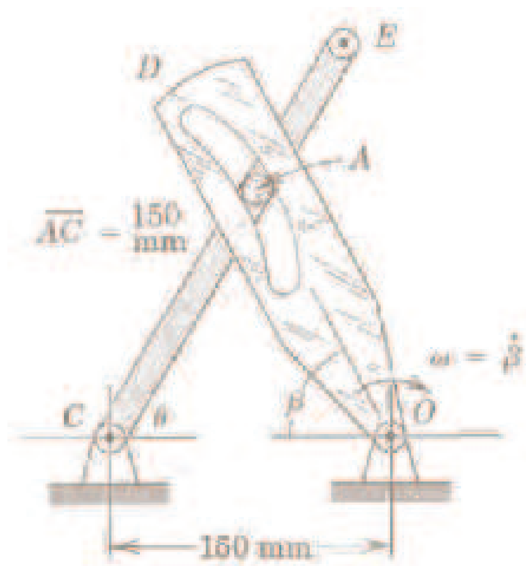


Figure 9.74: Example 9.36 ([3], pp. 410)

where, from the diagram,

$$v_P = 2 \times 0.15 = 0.3 \quad \text{m/s}$$

$$v_{\text{rel}} = v_P \tan 60^\circ = 0.5196 \quad \text{m/s}$$

$$v_A = v_P / \cos 60^\circ = 0.6 \quad \text{m/s}$$

Hence,

$$\omega_{AC} = v_A / \overline{AC} = 0.6/0.15 = 4 \quad \text{rad/s CW}$$

Acceleration analysis The acceleration relationship and its diagram, shown in fig. 9.75, can be established.

$[\mathbf{a}_A = \mathbf{a}_P + 2\boldsymbol{\omega}_{OA} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}]$

From the acceleration diagram, two unknowns are $(\mathbf{a}_A)_t$ and $(\mathbf{a}_{\text{rel}})_t$. Since the problem asks the angular acceleration of link EC, α_{EC}, only $(\mathbf{a}_A)_t$ is needed to be determined. Consider projecting the polygon onto the direction normal to $(\mathbf{a}_{\text{rel}})_t$ so its projection will be null, the following equation can be formulated:

$$(\mathbf{a}_A)_t \cos 60^\circ + 2.4 \cos 30^\circ + 0.9 = 2.0784 + 1.8$$

$[a_t = r\alpha]$ $$(\mathbf{a}_A)_t = 1.8 = 0.15\alpha_{EC}$$

$$\alpha_{EC} = 12 \quad \text{rad/s}^2 \quad \text{CCW}$$

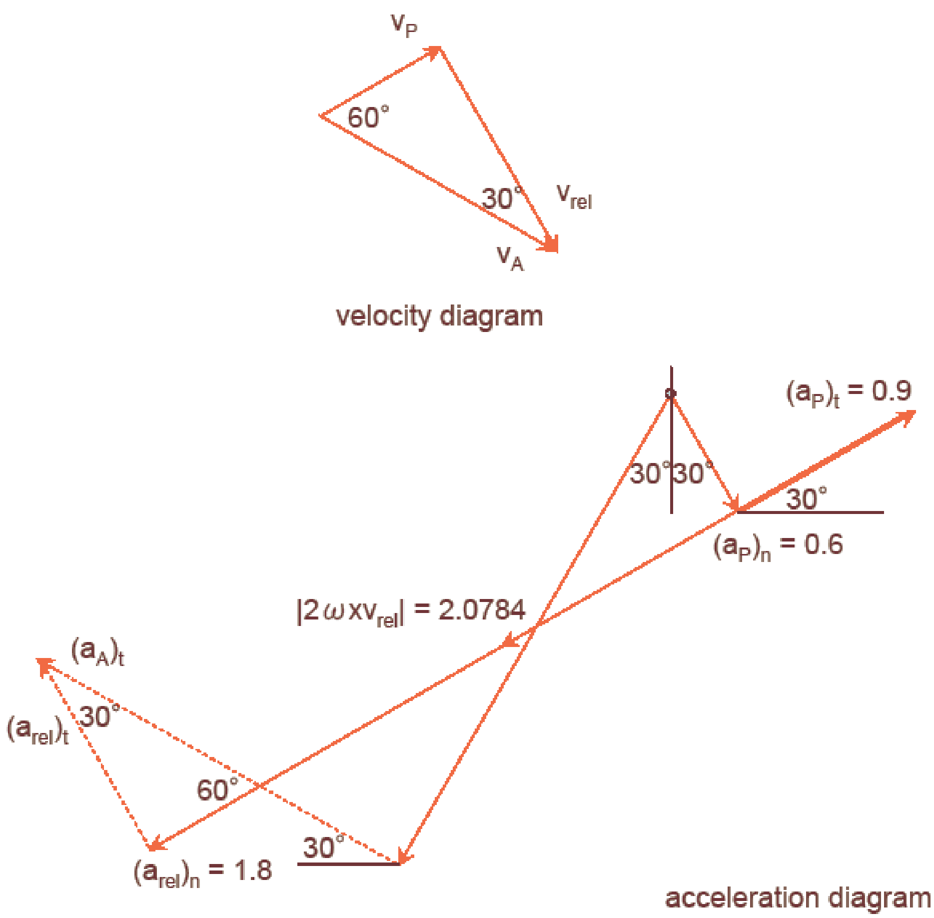


Figure 9.75: Solution to example 9.36

Bibliography

[1] J. L. Meriam and L. G. Kraige. *Engineering Mechanics*, volume 1, Statics. John Wiley & Sons, Inc., sixth edition, 2008. (document), 1.1, 1.2, 1.4, 2.4, 2.11, 2.5, 2.12, 2.18, 2.6, 2.8, 2.20, 2.9, 2.22, 2.28, 2.29, 2.30, 2.31, 2.14, 2.34, 2.36, 2.16, 2.17, 2.37, 2.18, 2.39, 2.19, 2.40, 2.20, 2.42, 2.43, 2.44, 2.45, 2.46, 2.22, 2.49, 2.51, 2.52, 2.53, 2.54, 2.55, 2.24, 2.56, 2.27, 2.59, 2.60, 2.61, 2.28, 2.62, 2.29, 2.63, 2.64, 2.31, 2.66, 3.1, 3.2, 3.1, 3.3, 3.4, 3.5, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.2, 3.14, 3.3, 3.16, 3.5, 3.21, 3.6, 3.24, 3.8, 3.29, 3.38, 3.39, 3.40, 3.12, 3.13, 3.43, 3.45, 3.14, 3.47, 3.15, 3.49, 3.18, 3.55, 3.20, 3.21, 3.59, 3.61, 3.22, 3.63, 4.1, 4.2, 4.1, 4.3, 4.2, 4.4, 4.3, 4.5, 4.6, 4.7, 4.8, 4.4, 4.9, 4.10, 4.5, 4.11, 4.6, 4.12, 4.13, 4.7, 4.15, 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.1, 5.7, 5.8, 5.9, 5.10, 5.11, 5.18, 5.23, 5.24, 5.8, 5.26, 5.27, 5.28, 5.29, 5.30, 5.31, 5.32, 5.33, 5.34, 5.9, 5.35, 5.11, 5.39, 5.13, 5.14, 5.43, 5.45, 5.15, 5.47, 5.16, 5.49

[2] J. L. Meriam and L. G. Kraige. *Engineering Mechanics*, volume 1, Statics. John Wiley & Sons, Inc., third edition, 1993. (document), 2.5, 2.1, 2.7, 2.2, 2.3, 2.9, 2.10, 2.24, 2.11, 2.25, 2.12, 2.32, 2.13, 2.33, 2.15, 2.35, 2.21, 2.47, 2.23, 2.50, 2.25, 2.57, 2.26, 2.58, 2.30, 2.65, 3.4, 3.19, 3.7, 3.27, 3.9, 3.32, 3.10, 3.34, 3.11, 3.41, 3.16, 3.51, 3.17, 3.53, 3.19, 3.57, 4.8, 4.9, 4.17, 4.19, 4.10, 4.21, 5.2, 5.12, 5.3, 5.14, 5.4, 5.16, 5.5, 5.19, 5.6, 5.21, 5.25, 5.7, 5.10, 5.37, 5.12, 5.41

[3] J. L. Meriam and L. G. Kraige. *Engineering Mechanics*, volume 2, Dynamics. John Wiley & Sons, Inc., sixth edition, 2008. (document), 7.1, 7.2, 7.3, 7.1, 7.4, 7.3, 7.8, 7.5, 7.11, 7.14, 7.15, 7.16, 7.17, 7.18, 7.9, 7.21, 7.11, 7.24, 7.12, 7.25, 7.27, 7.28, 7.29, 7.14, 7.30, 7.15, 7.31, 7.18, 7.37, 7.38, 7.39, 7.40, 7.20, 7.21, 7.41, 7.43, 7.23, 7.24, 7.46, 7.47, 7.27, 7.52, 7.57, 7.30, 7.58, 7.31, 7.32, 7.59, 7.33, 7.61, 7.62, 8.1, 8.1, 8.2, 8.3, 8.3, 8.5, 8.5, 8.6, 8.9, 8.11, 8.7, 8.13, 8.8, 8.15, 8.11, 8.22, 8.12, 8.24, 8.14, 8.28, 8.15, 8.30, 8.18, 8.37, 8.21, 8.43, 9.1, 9.2, 9.3, 9.4, 9.5, 9.1, 9.2, 9.7, 9.3, 9.4, 9.8, 9.9, 9.10, 9.11, 9.5, 9.12, 9.8, 9.17, 9.9, 9.19, 9.11, 9.21, 9.12, 9.22, 9.23, 9.24, 9.25, 9.13, 9.26, 9.14, 9.28, 9.30, 9.15, 9.16, 9.32, 9.18, 9.36, 9.19, 9.38, 9.41, 9.22, 9.23, 9.43, 9.45, 9.48, 9.49, 9.27, 9.53, 9.28, 9.55, 9.30, 9.59, 9.61, 9.62, 9.63, 9.64, 9.31, 9.65, 9.33, 9.69, 9.34, 9.70, 9.36, 9.74

[4] J. L. Meriam and L. G. Kraige. *Engineering Mechanics*, volume 2, Dynamics. John Wiley & Sons, Inc., third edition, 1993. (document), 7.2, 7.6, 7.4, 7.10, 7.6, 7.7, 7.12, 7.13, 7.8, 7.19, 7.10, 7.23, 7.13, 7.26, 7.16, 7.17, 7.33, 7.35, 7.22, 7.45, 7.25, 7.48, 7.26, 7.50, 7.28, 7.54, 7.29, 7.56, 8.7, 8.4, 8.9, 8.17, 8.10, 8.20, 8.13, 8.26, 8.16, 8.17, 8.32, 8.35, 8.19, 8.39, 8.20, 8.41, 8.22, 8.23, 8.45, 8.47, 9.6, 9.14, 9.7, 9.16, 9.10, 9.20, 9.17, 9.34, 9.20, 9.39, 9.21, 9.42, 9.24, 9.47, 9.25, 9.50, 9.26, 9.51, 9.29, 9.57, 9.32, 9.67, 9.35, 9.72

[5] R. C. Hibbeler. *Engineering Mechanics, Statics*. Prentice-Hall. (document)

[6] R. C. Hibbeler. *Engineering Mechanics, Dynamics*. Prentice-Hall. (document)